Satz: Ist M_{inc} ein inklusionsmaximales Matching und M_{max} ein kardinalitäts-maximales Matching, so gilt $|M_{inc}| \geq |M_{max}| \;/\; 2.$

Proof

We consider edges from the exclusive disjunction of M_{max} and M_{inc}, i.e. $e \in M_{\text{max}} \oplus M_{\text{inc}} = (M_{\text{max}} \setminus M_{\text{inc}}) \cup (M_{\text{inc}} \setminus M_{\text{max}})$.

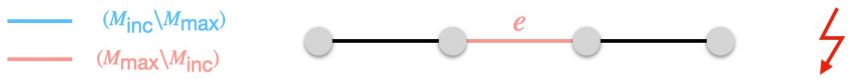
(i) Each edge in $(M_{\text{inc}} \setminus M_{\text{max}})$ can be adjacent to at most two edges in $(M_{\text{max}} \setminus M_{\text{inc}})$.

Clearly, any other edge in $(M_{\text{max}} \setminus M_{\text{inc}})$ adjacent to e would contradict the fact that M_{max} is a matching.

do we already know $|M_{max} \setminus M_{inc}| \leq 2 \cdot |M_{inc} \setminus M_{max}|$

no! we also need to rule out "isolated edges" in $M_{max} \setminus M_{inc}$.

(ii) Each edge in $(M_{\text{max}} \setminus M_{\text{inc}})$ is adjacent to an edge in $(M_{\text{inc}} \setminus M_{\text{max}})$.



If e was not adjacent to an edge in $(M_{\text{inc}} \setminus M_{\text{max}})$, then we could add e to M_{inc}. Contradiction.

(i) + (ii) $\Rightarrow$ $|M_{max} \setminus M_{inc}| \leq 2 \cdot |M_{inc} \setminus M_{max}|$

thus:

$$\begin{aligned} |M_{max}| &= |M_{max} \cap M_{inc}| + |M_{max} \setminus M_{inc}| \\ &\leq 2 \cdot |M_{inc} \cap M_{max}| + 2\,|M_{inc} \setminus M_{max}| \\ &= 2 \cdot |M_{inc}| \end{aligned}$$