Algorithms and Probability

Week 3

2025/03/06 — Georg Hasebe

Satz von Hall

Satz 1.52 (Satz von Hall, Heiratssatz). Für einen bipartiten Graphen $G = (A \uplus B, E)$ gibt es genau dann ein Matching M der Kardinalität $|M| = |A|$, wenn gilt

$$|N(X)| \geq |X| \quad \text{für alle } X \subseteq A. \tag{1.1}$$

Note the notation $N(X)$ is defined as: $N(X) := \bigcup_{v \in X} N(v)$.

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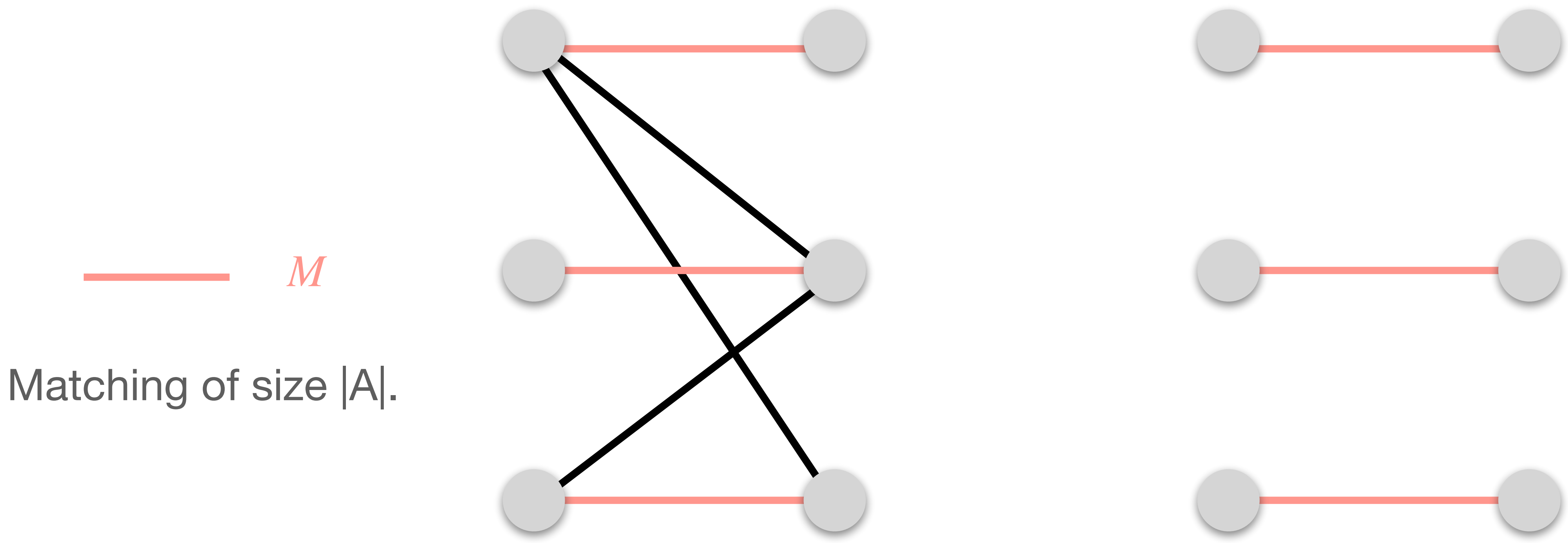
$$|N(X)| \geq |X| \quad \text{für alle } X \subseteq A. \tag{1.1}$$

Proof

($\Longrightarrow$) Assume $G = (A \uplus B, E)$ has a matching M of size $|M| = |A|$.

In the subgraph $H = (A \uplus B, M)$, every subset $X \subseteq A$ has exactly $|X|$ neighbors (i.e. $|N(X)| = |X|$) by definition of a matching.

Since $M \subseteq E$ we have that $|N(X)| \geq |X|$ for all $X \subseteq A$.



$G = (A \uplus B, E)$ $\qquad$ $H = (A \uplus B, M)$

Satz 1.52 (Satz von Hall, Heiratssatz). Für einen bipartiten Graphen $G = (A \uplus B, E)$ gibt es genau dann ein Matching M der Kardinalität $|M| = |A|$, wenn gilt

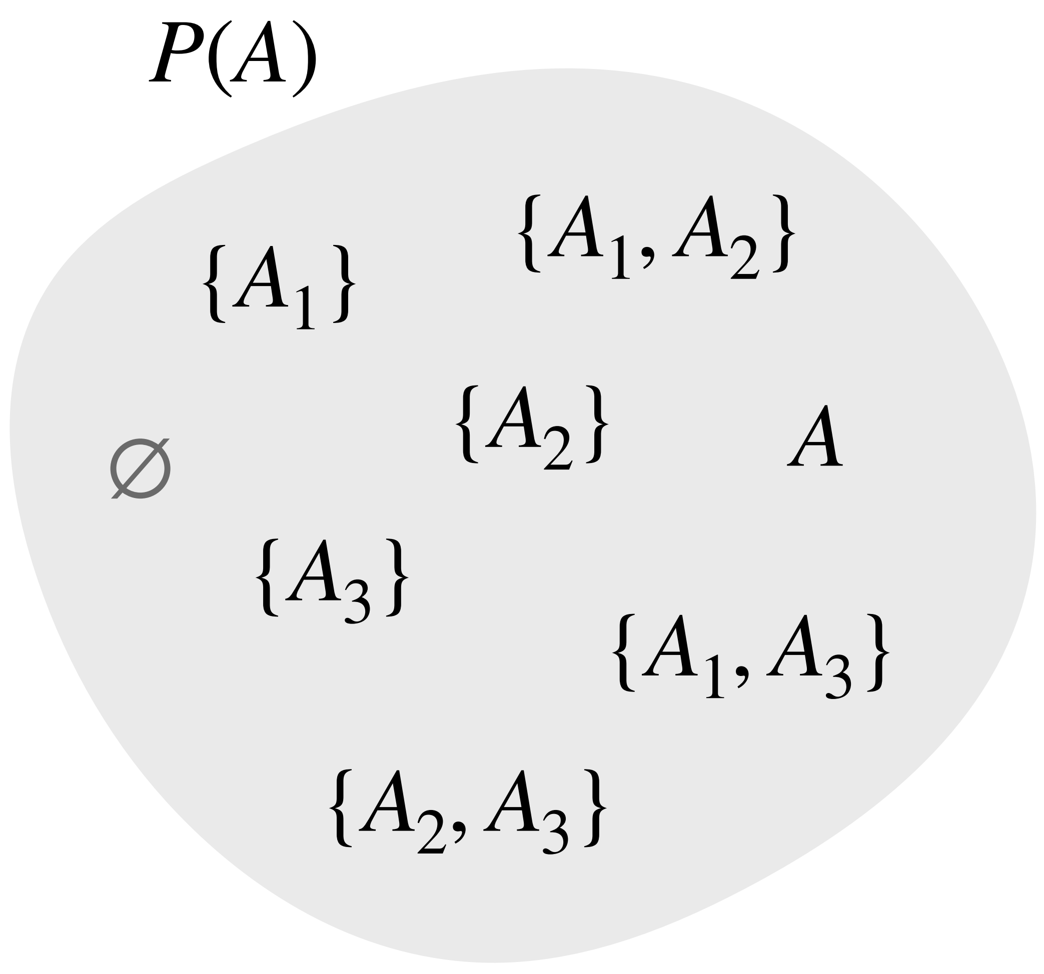
$$|N(X)| \geq |X| \quad \text{für alle } X \subseteq A. \tag{1.1}$$

Proof

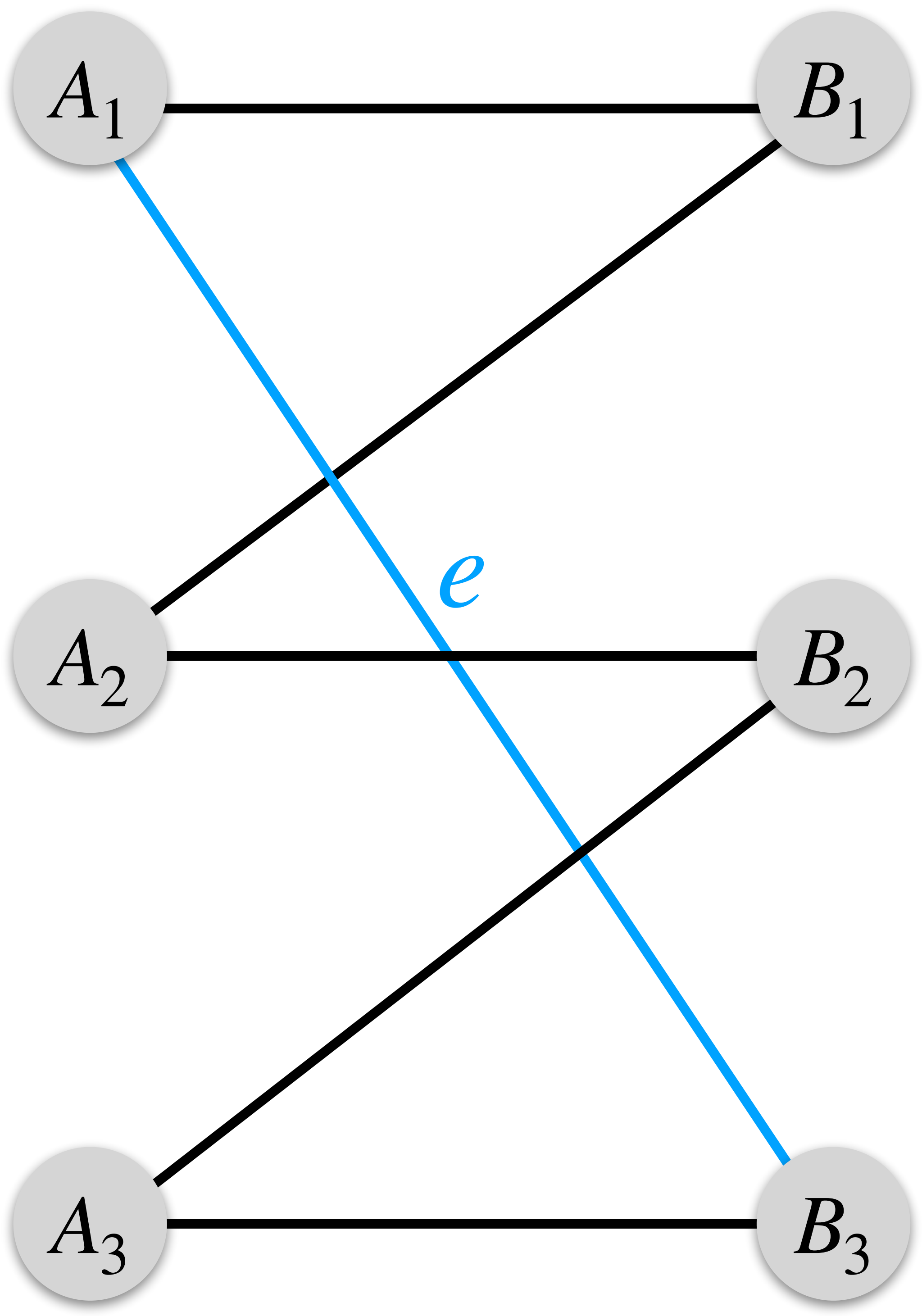
($\Longleftarrow$) on the blackboard.

Proof

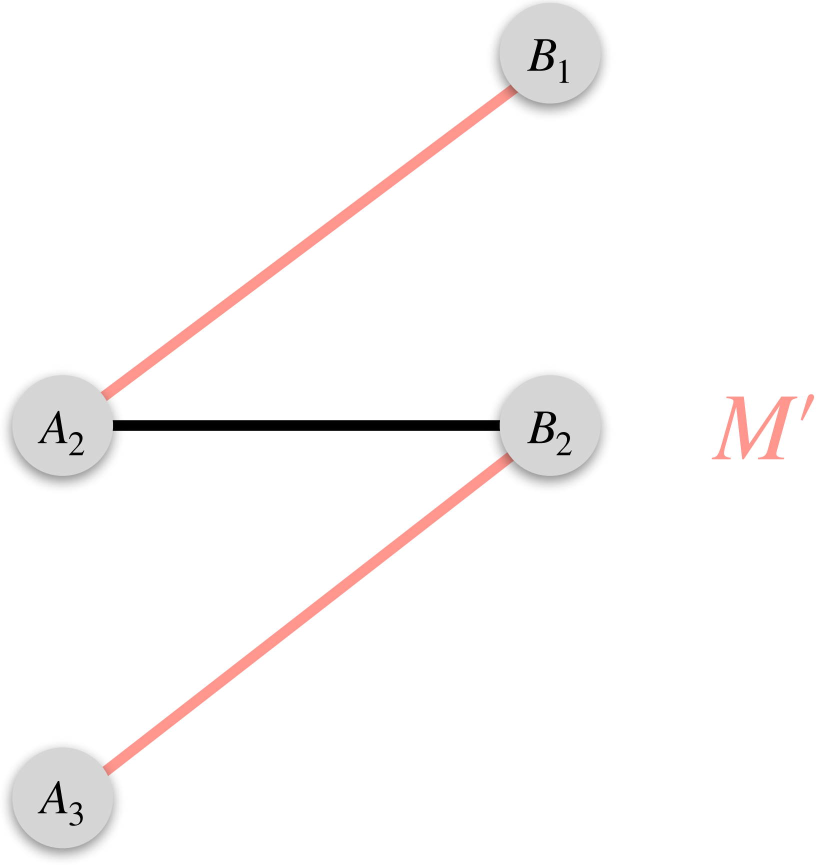
$P(A)$ is the powerset of A.



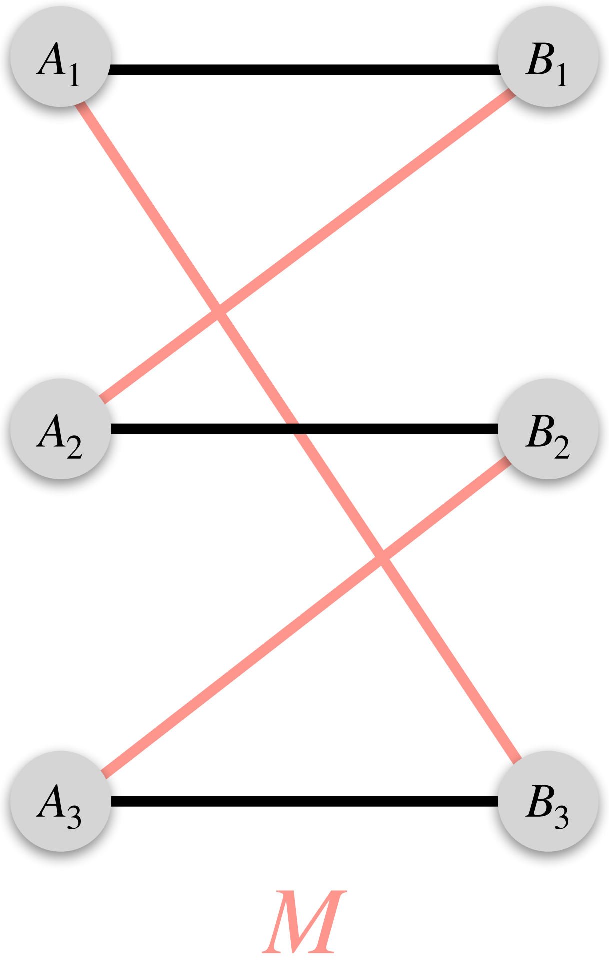
For all $X \in P(A)$ we assume that $|N(X)| > |X|$. You can verify this yourself for the example.



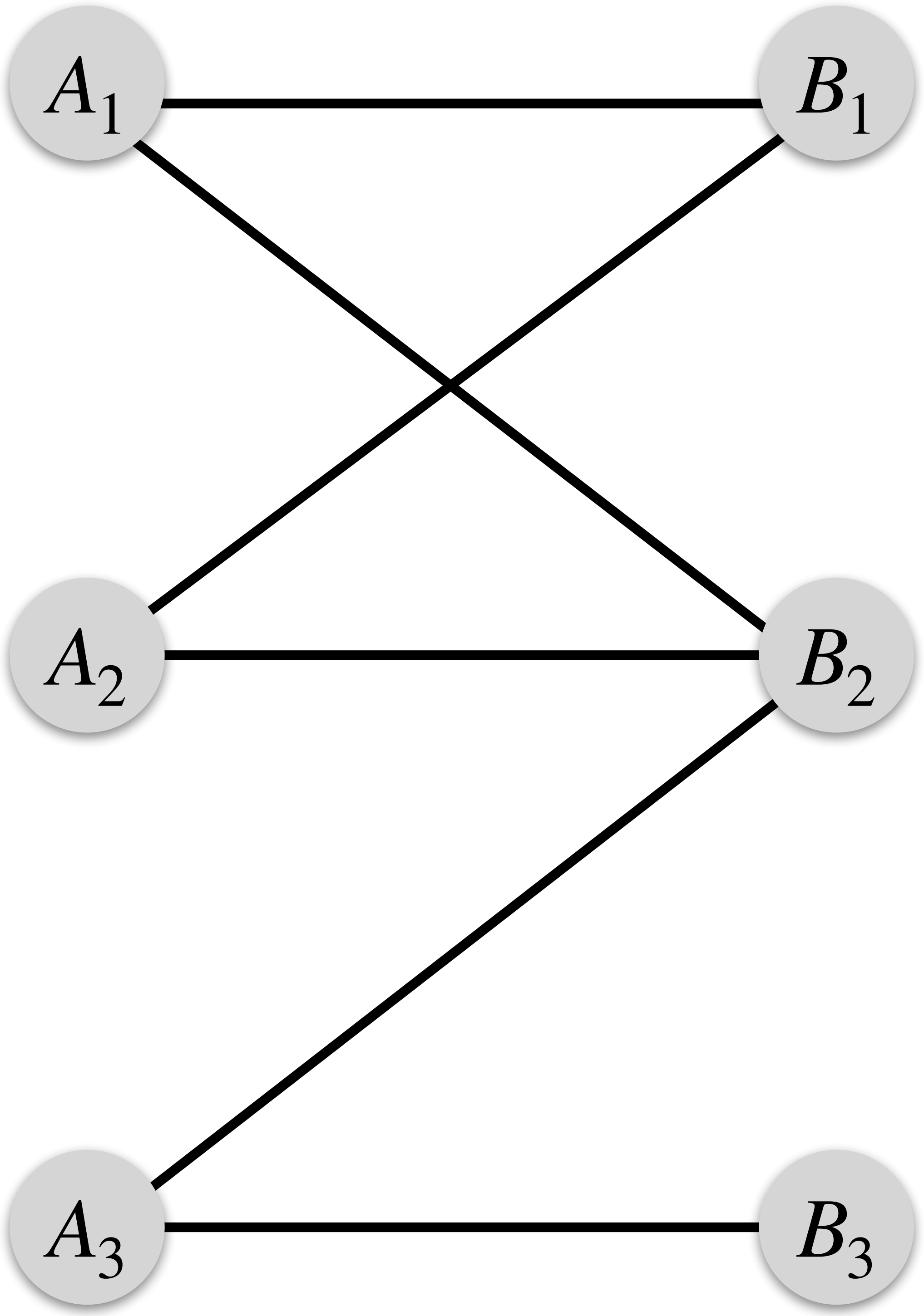
remove $e = \{A_1, B_3\}$



extend by e to get M of size $|M| = |A|$.

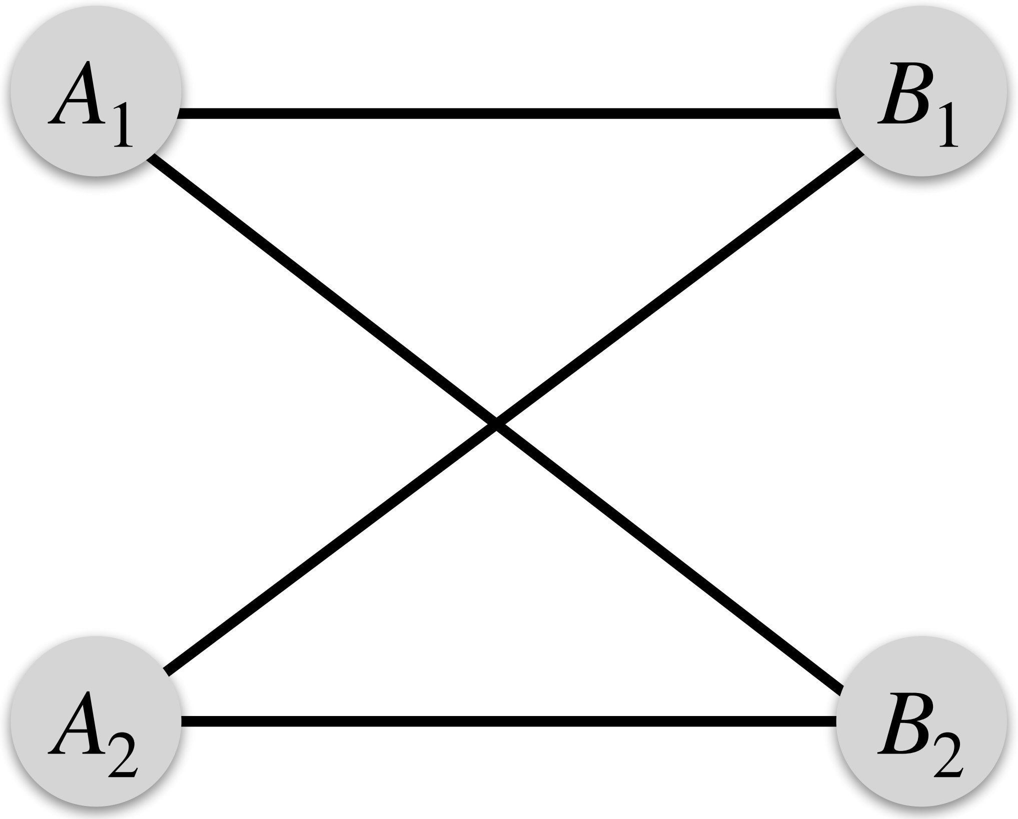


Proof



$$G = (A \uplus B, E)$$

$$|X_0| = |\{A_1, A_2\}| = |\{B_1, B_2\}| = |N(X_0)|$$



$$G' = G[X_0 \uplus N(X_0)]$$

For G' condition (1.1) holds.

$$G'' = G[A \backslash X_0 \uplus B \backslash N(X_0)]$$

What about G''?

Proof

We can find a perfect matchings M' and M'' in G' and G'' (since condition (1.1) holds).

Then $M = M' \cup M''$ is a perfect matching in G.