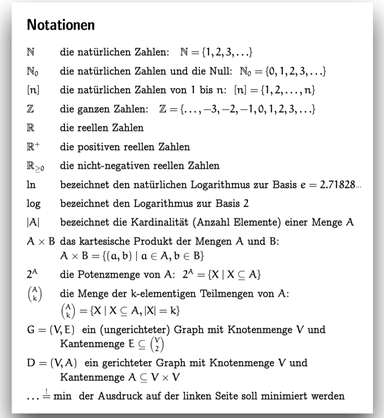
Algorithm and Data Structures Recap

2025/02/20 — Georg Hasebe

Algorithms and Data Structures Recap

Math notation and terminology

- $\mathbb{N} = \{1,2,3,\ldots\}$, $\mathbb{N}_0 = \{0,1,2,3,\ldots\}$, $[n] = \{1,2,3,\ldots,n\}$
- $\ln$ logarithm to the base e, $\log$ logarithm to the base 2
- $|X|$ cardinality of X (number of elements in X)
- $2^X = \{Y \mid Y \subseteq X\}$ (power set of X)
- $\binom{X}{k} = \{Y \mid Y \subseteq X, |Y| = k\}$ (the set of subsets of X of cardinality k)



Notationen

$\mathbb{N}$	die natürlichen Zahlen: $\mathbb{N} = \{1, 2, 3, \ldots\}$
$\mathbb{N}_0$	die natürlichen Zahlen und die Null: $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$
$[n]$	die natürlichen Zahlen von 1 bis n: $[n] = \{1, 2, \ldots, n\}$
$\mathbb{Z}$	die ganzen Zahlen: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
$\mathbb{R}$	die reellen Zahlen
$\mathbb{R}^+$	die positiven reellen Zahlen
$\mathbb{R}_{\geq 0}$	die nicht-negativen reellen Zahlen
ln	bezeichnet den natürlichen Logarithmus zur Basis $e = 2.71828\ldots$
log	bezeichnet den Logarithmus zur Basis 2
$\lvert A\rvert$	bezeichnet die Kardinalität (Anzahl Elemente) einer Menge A
$A \times B$	das kartesische Produkt der Mengen A und B: $A \times B = \{(a, b) \mid a \in A, b \in B\}$
2^A	die Potenzmenge von A: $2^A = \{X \mid X \subseteq A\}$
$\binom{A}{k}$	die Menge der k-elementigen Teilmengen von A: $\binom{A}{k} = \{X \mid X \subseteq A, \lvert X\rvert = k\}$
$G = (V, E)$	ein (ungerichteter) Graph mit Knotenmenge V und Kantenmenge $E \subseteq \binom{V}{2}$
$D = (V, A)$	ein gerichteter Graph mit Knotenmenge V und Kantenmenge $A \subseteq V \times V$
$\ldots \doteq \min$	der Ausdruck auf der linken Seite soll minimiert werden

siehe Skript.

Examples

$$[3] = \{1,2,3\}$$

$$X = \{1,2\}$$

$$|X| = 2$$

$$2^X = \{\varnothing, \{1\}, \{2\}, \{1,2\}\}$$

$$A = \{1,2,3,4\}$$

$$\binom{A}{2} = \{\{1,2\}, \{2,3\}, \{3,4\}, \{1,3\}, \{1,4\}, \{2,4\}\}$$

(*) The notation $\binom{A}{k}$ also denotes the binomial coefficient, where A is not a set but an integer.

Algorithms and Data Structures Recap

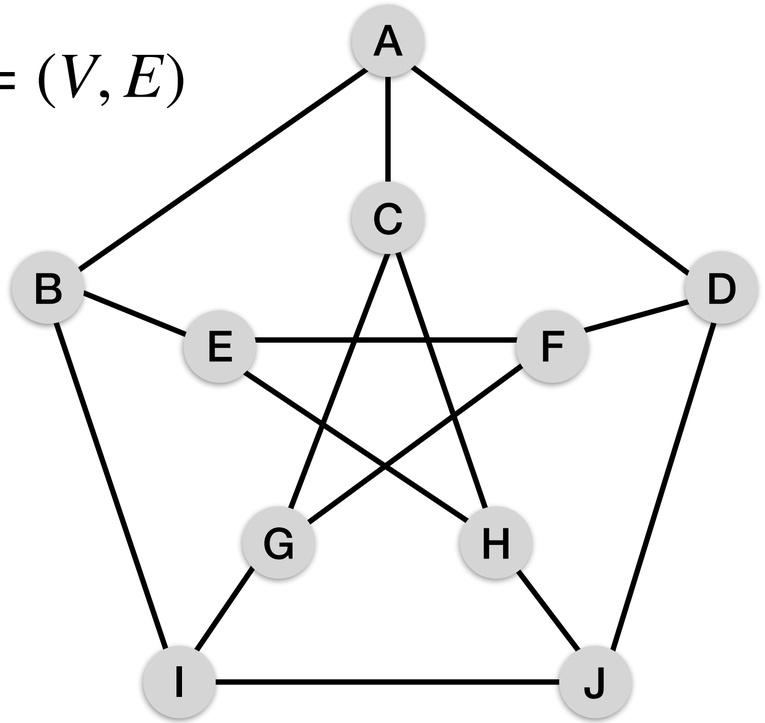
Graph notation and terminology

- $G = (V, E),\ |V|,\ |E| \ldots$
- $N_G(v) := \{u \in V \mid \{u, v\} \in E\}$ (Nachbarschaft von v in G)
- $G[X], \quad X \subseteq V$ (induced subgraph with vertex set X and edges of E that have both endpoints in X)
- $\deg(v), \deg_{\text{in}}(v), \deg_{\text{out}}(v), \deg^{-}(v), \deg^{+}(v)$ (number of edges: in/- for “eingehend”, out/+ for “ausgehend”)

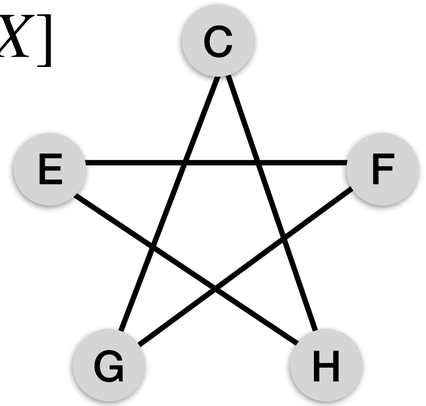
(*) We often shorten $N_G(v)$ to $N(v)$ when it’s clear what graph G is meant.

Examples

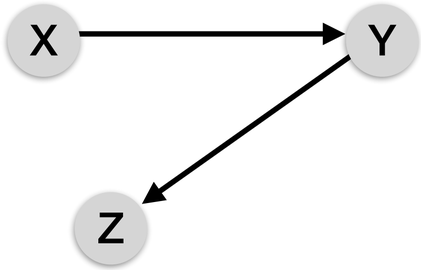
$G = (V, E)$



$G[X]$



$X = \{C, E, F, G, H\}$



$N_G(C) = \{A, G, H\}$

$\deg(F) = 2$

$\deg^+(X) = \deg_{\text{out}}(X) = 1$

$\deg^-(X) = \deg_{\text{in}}(X) = 0$

Algorithms and Data Structures Recap

Notation and terminology

- A sequence of vertices $< v_1, v_2, \ldots, v_k >$, such that v_i and v_{i+1} are connected by an edge
  - Is a walk (Weg) of length $k - 1$.
  - Is a closed walk (geschlossener Weg), if $v_1 = v_k$.
  - Is a path (Pfad), if all vertices are distinct.
  - Is a cycle (Kreis), if all vertices are distinct and $v_1 = v_k$.

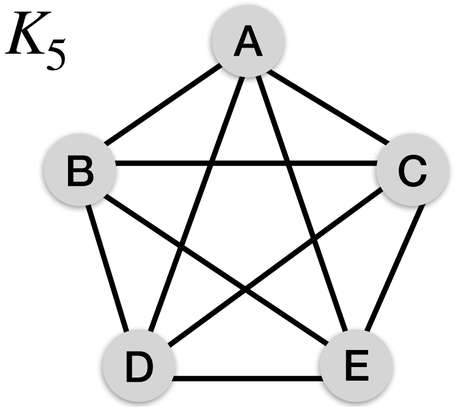
Algorithms and Data Structures Recap

Important graphs

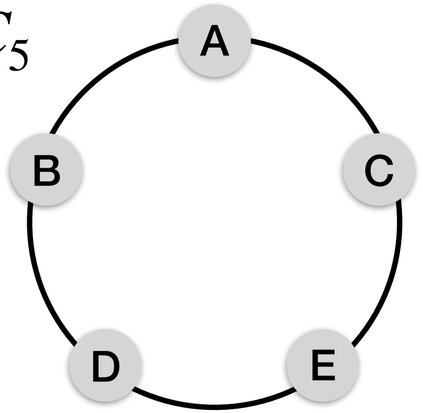
- K_n denotes the *complete graph* on n vertices, where every pair of different vertices is connected by an edge.
- C_n denotes the *cycle graph* on n vertices.
- P_n denotes the *path graph* on n vertices.

Examples

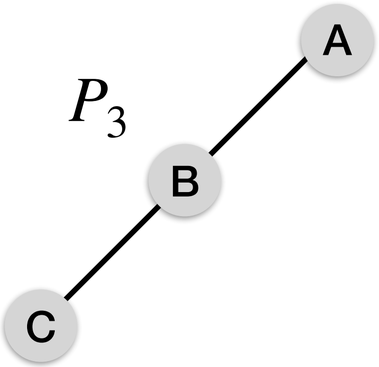
K_5



C_5



P_3



Algorithms and Data Structures Recap

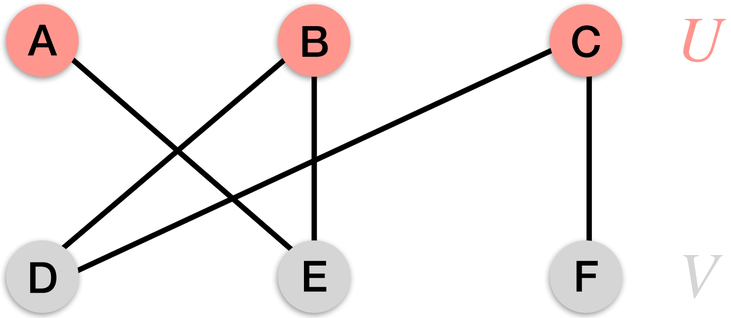
Bipartite graphs

- A *bipartite graph* is a graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V.
- We usually write $G = (U \uplus V, E)$ for a bipartite graph G.

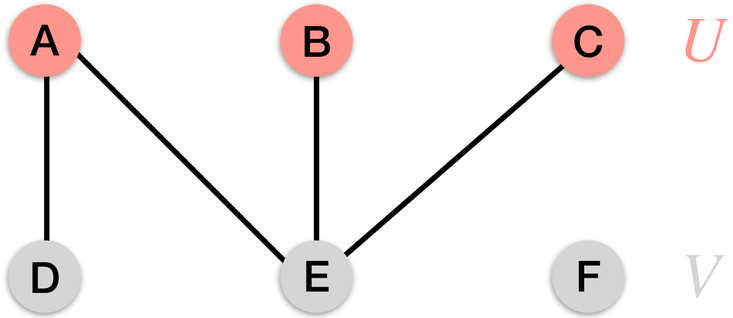
(*) disjoint simply means that $U \cap V = \emptyset$
(**) the symbol $\uplus$ is often used to denote a disjoint union of sets, i.e. $U \cup V$ such that $U \cap V = \emptyset$

Example

$$G = (U \uplus V, E)$$

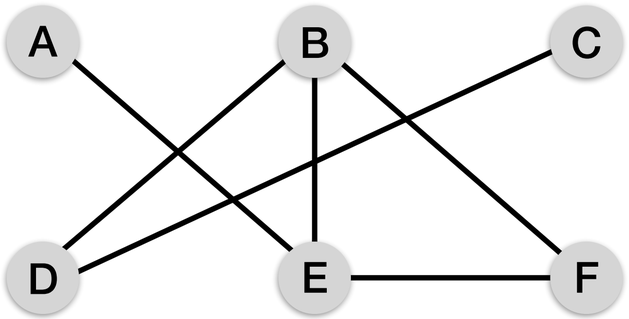


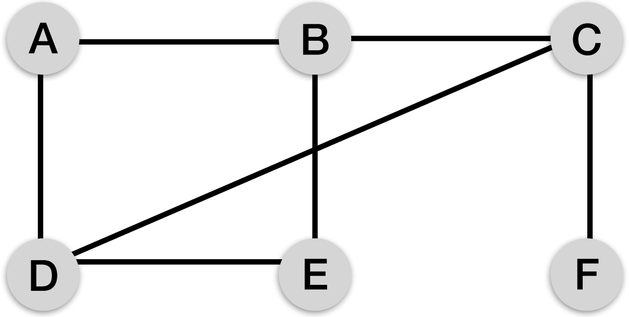
$$\tilde{G} = (U \uplus V, \tilde{E})$$

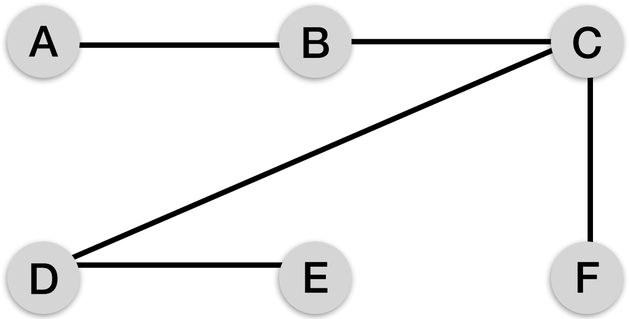


(*) A *bipartite graph* must not be connected, hence $\tilde{G}$ is also *bipartite*.

Bipartite?

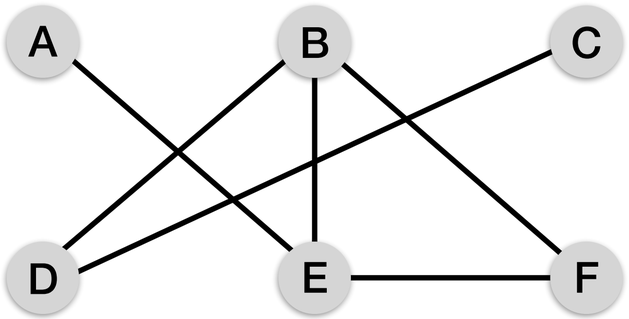






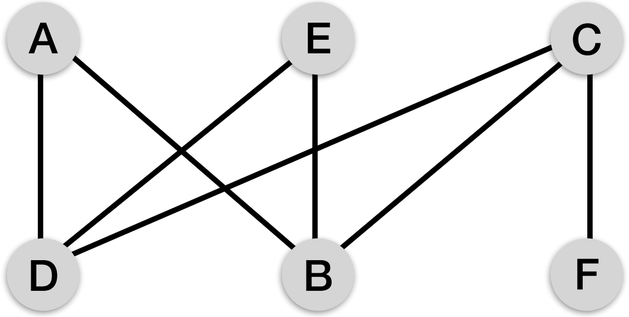
Bipartite?

(1)



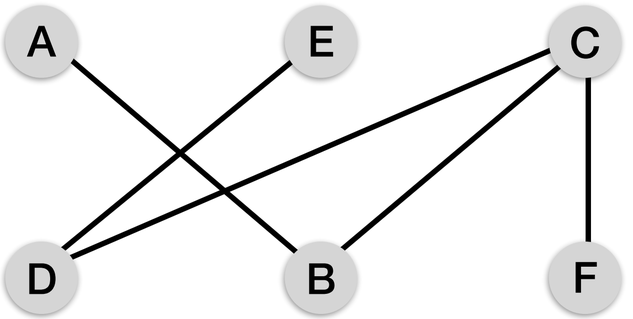
not bipartite

(2)



bipartite

(3)



bipartite

(*) The left graph contains an odd-length cycle $< B, E, F >$. A bipartite graph cannot contain any odd-length cycles!
(**) Proof left as an exercise (or click here).

Algorithms and Data Structures Recap

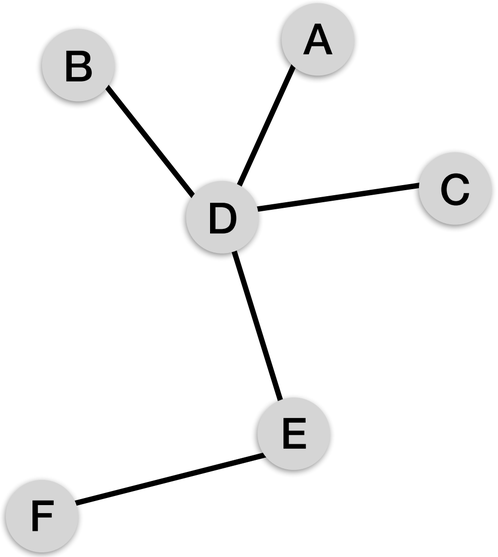
Trees

A *tree* is an undirected graph G that satisfies any of the following equivalent conditions:

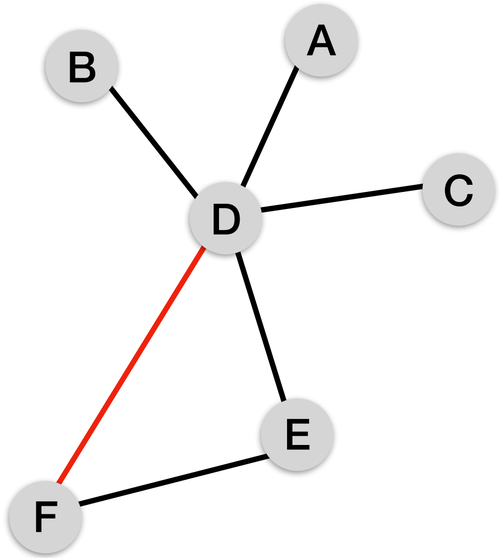
- G is connected and acyclic.
- G is acyclic and a simple cycle is formed if any edge is added.
- G is connected and has $n - 1$ vertices.
- G is connected, but would be disconnected if single edge is removed.
- Any two vertices in G can be connected by a unique path.

(*) for the third bullet point we assume that G has a finite amount of vertices.

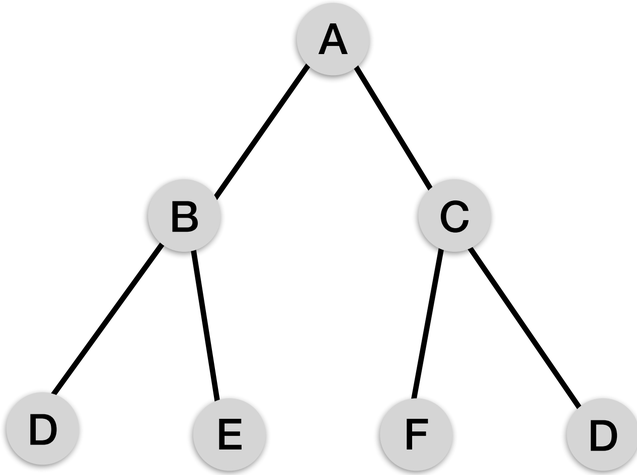
Examples I



$T_1 = (V, E)$



$G = (V, E)$

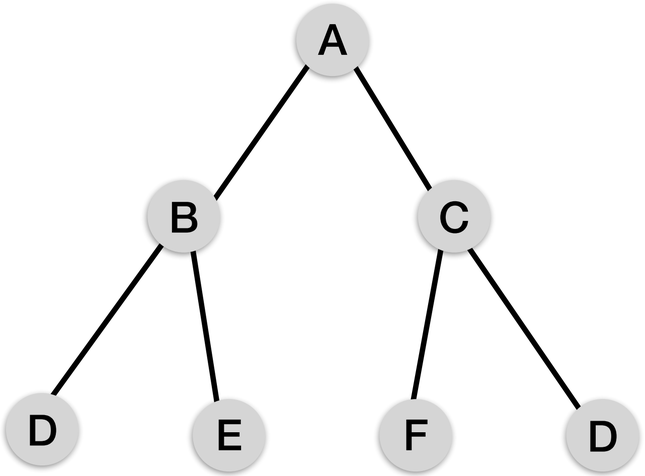


$T_2 = (V, E)$

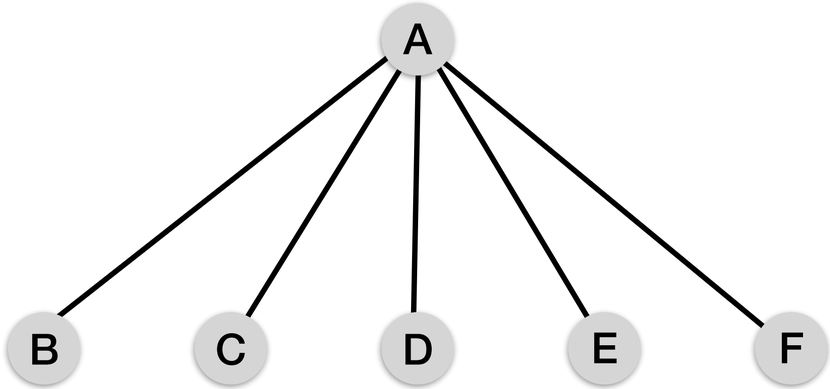
(*) T_1 and T_2 are trees, T_2 is a *binary tree* since any node has at most two children.

(**) G is not a tree since (1) it has a cycle (2) it has $n = 6$ edges (instead of $n - 1 = 5$) (3) it would not be disconnected if the red edge were to be removed (4) D and F are connected by two paths.

Examples II



$T_1 = (V, E)$



$T_2 = (V, E)$

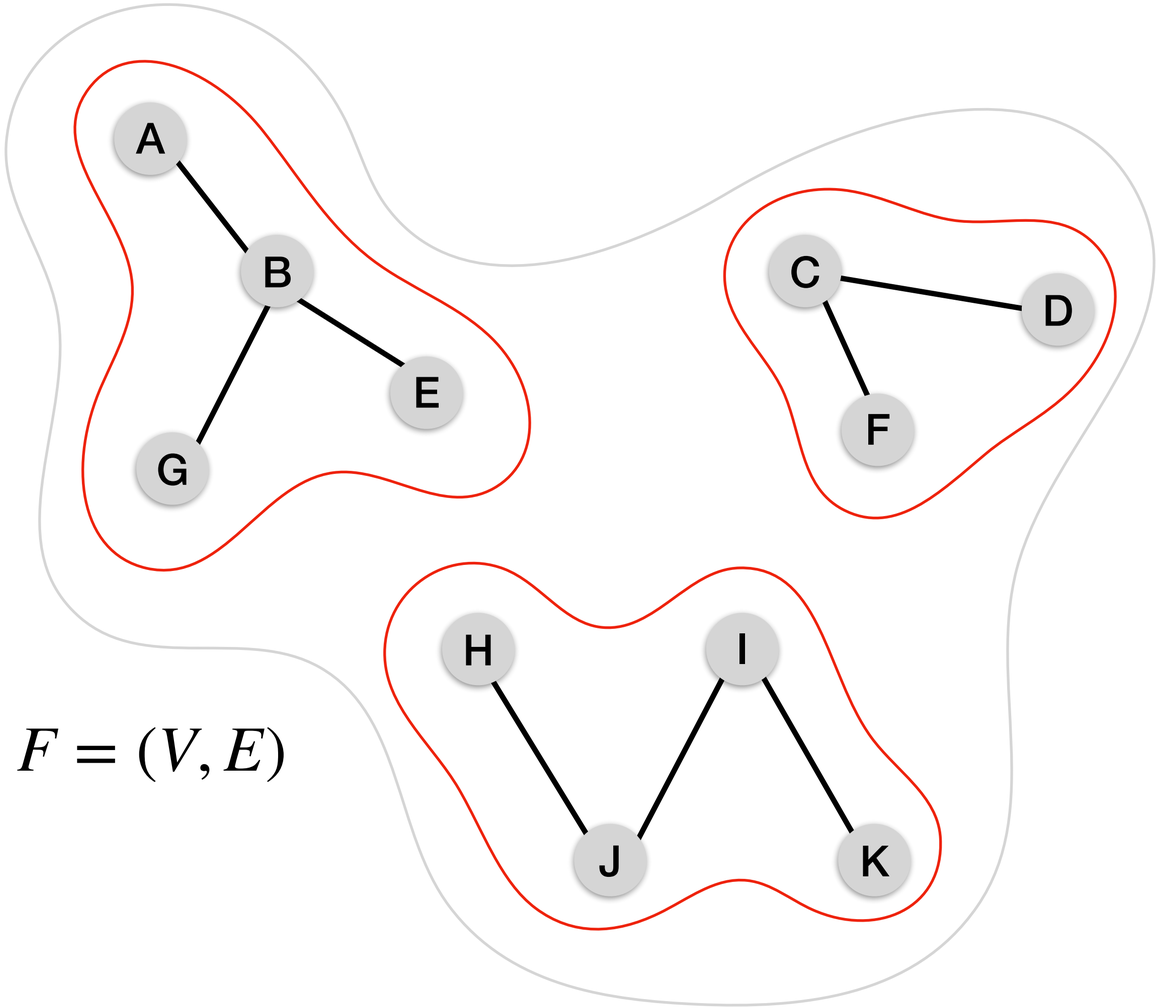
(*) Students sometimes think of only binary trees (e.g. T_1) when reasoning about trees. However, T_2 is a tree as well!

Algorithms and Data Structures Recap

Trees, connected components and forests

- A *leaf* is a vertex in a tree of degree one.
- A *connected component* of an undirected graph is a connected subgraph that is not part of any larger connected subgraph. The connected components of any graph partition its vertices into disjoint sets.
- A *forest* is an undirected and acyclic graph whose connected components are trees. In other words, it consists of a disjoint union of trees.

Examples



$F = (V, E)$

(*) F is a *forest*. The three *connected components* of F are in red blobs, each of the graphs contained in one of the red blobs is a tree.

(**) The subgraph on the vertices $\{C, D, F\}$ is a tree, and both F and D is a *leaf*.

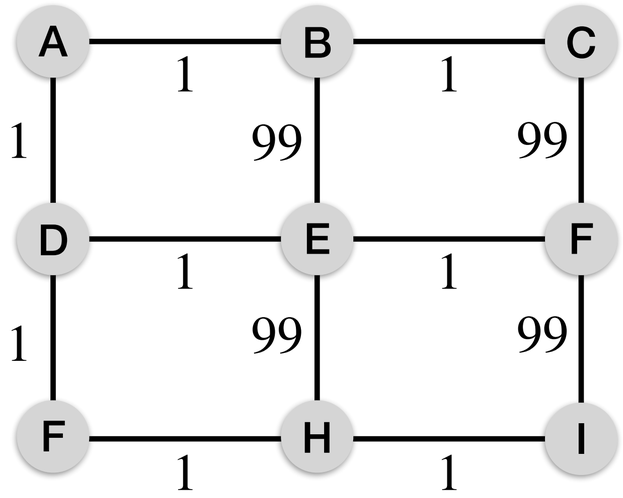
Algorithms and Data Structures Recap

Spanning trees

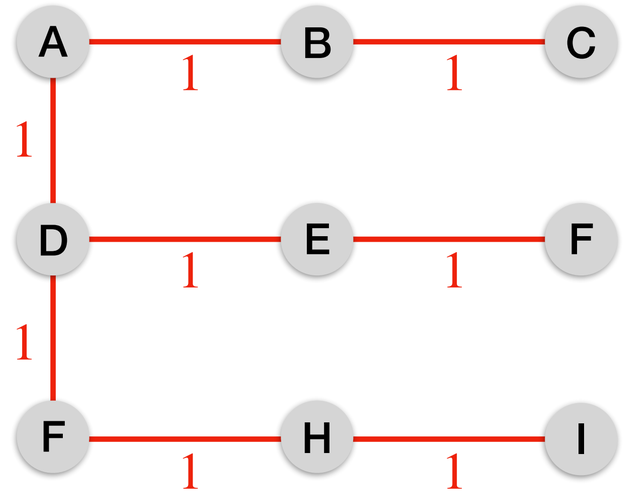
- A *spanning tree* T of an undirected graph G is a subgraph that is a tree which includes all the vertices of G.
- A *minimum spanning tree* (MST) is a spanning tree with the minimum possible sum of edge weights.

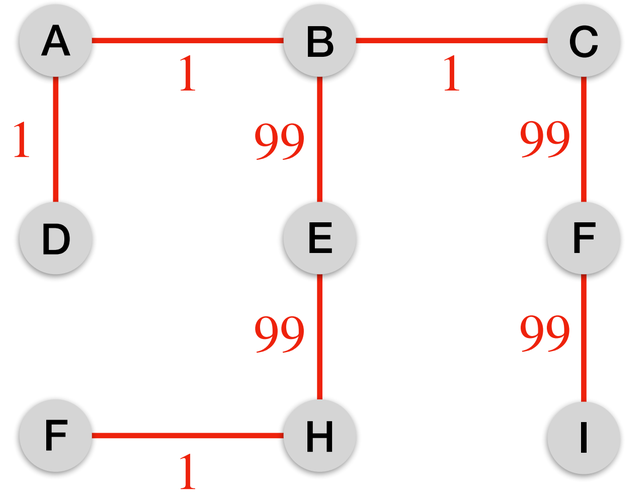
Examples

$G = (V, E)$



$T_1 = (V, E_2)$





$T_2 = (V, E_2)$

(*) Both T_1 and T_2 are *spanning trees* of G.

(**) T_1 is the unique *minimum spanning tree* of G of total weight 8.

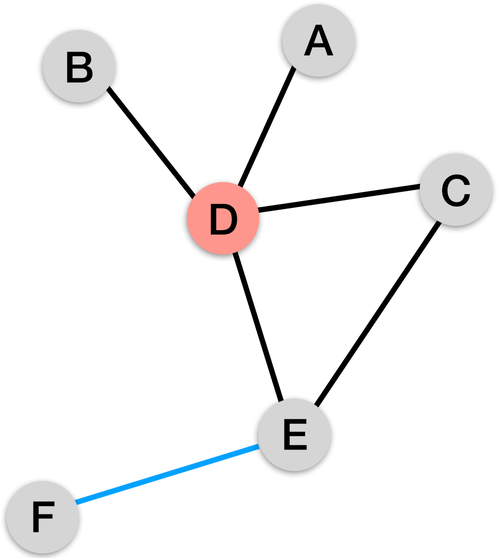
(***) G is the so called *grid graph*.

Algorithms and Data Structures Recap

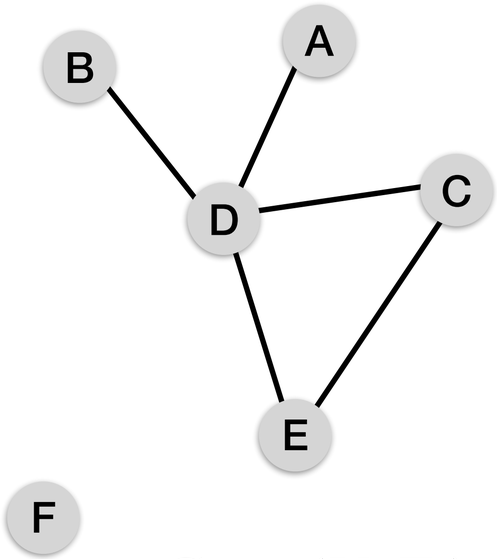
Cut vertices and bridges

- A *cut vertex* is any vertex whose removal increases the number of connected components.
- A *bridge* is any edge whose removal increases the number of connected components.

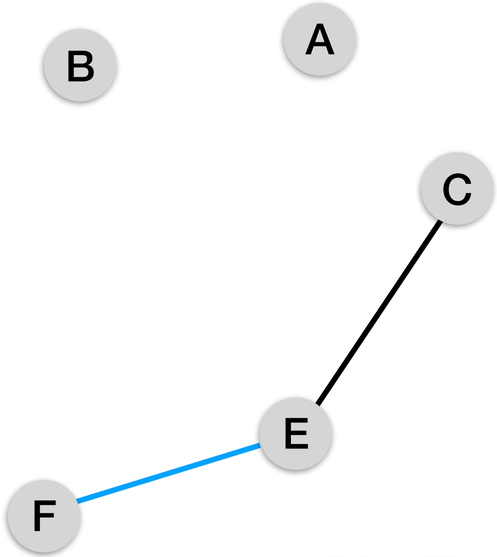
Examples



$G = (V, E)$



$G_1 = (V, E_1)$



$G_2 = (V_2, E_2)$

(*) The blue edge is a *bridge*. It's removal increases the number of connected components in G (which is originally one) to two (see graph G_1).
(**) The reddish vertex is a *cut vertex*. It's removal increases the number of connected components in G (which is originally one) to three (see in graph G_2).
(***) Note there are multiple cut vertices and bridges in G. The vertex C is *not* a cut vertex and the edge $\{E, C\}$ is *not* a bridge.

AnD Recap

BFS-Visit-Iterative(G, v)

```
1  Q ← ∅
2  Markiere v als aktiv
3  Enqueue(Q, v)
4  while Q ≠ ∅ do
5      w ← Dequeue(Q)
6      Markiere w als besucht
7      for each (w, x) ∈ E do
8          if x nicht aktiv und x noch nicht besucht then
9              Markiere x als aktiv
10             Enqueue(Q, x)
```

Queue

Stack

DFS-Visit-Iterative(G, v)

```
1  S ← ∅
2  Push(S, v)
3  while S ≠ ∅ do
4      w ← Pop(S)
5      if w noch nicht besucht then
6          Markiere w als besucht
7          for each (w, x) ∈ E in reverse order do
8              if x noch nicht besucht then
9                  Push(S, x)
```

Solving graph exercises?

- Questions that I often received in my AnD exercise classes are
  - How do I solve graph exercises?
  - How rigorous/long should my proof be?
- Of course there is not a single “one size fits all” solution, but what I always tell them is, the more you know about graphs (e.g. properties of a graph, useful lemmas, equivalences…), the better! Let’s go through an example.

For any graph $G = (V, E)$ we have: the number of vertices with odd degree is even.

For any graph $G = (V, E)$ we have: the number of vertices with odd degree is even.

Theorem 1.2. For every graph $G = (V, E)$ we have $\sum_{v \in V} \deg(v) = 2 \cdot |E|$.

(*) knowing Theorem 1.2. things get a lot easier!

For any graph $G = (V, E)$ we have: the number of vertices with odd degree is even.

Theorem 1.2. For every graph $G = (V, E)$ we have $\sum_{v \in V} \deg(v) = 2 \cdot |E|$.

Proof. Let V_e and V_o be the vertex set of the vertices with even and odd degree.

We have $\sum_{v \in V} \deg(v) = \sum_{v \in V_e} \deg(v) + \sum_{v \in V_o} \deg(v)$.

The sum of all even degrees is even. The sum of k odd numbers is even if and only if k is even. Using Theorem 1.2. we conclude that $|V_o|$ is even.

(*) the first gray box is corollary 1.3. in the script.

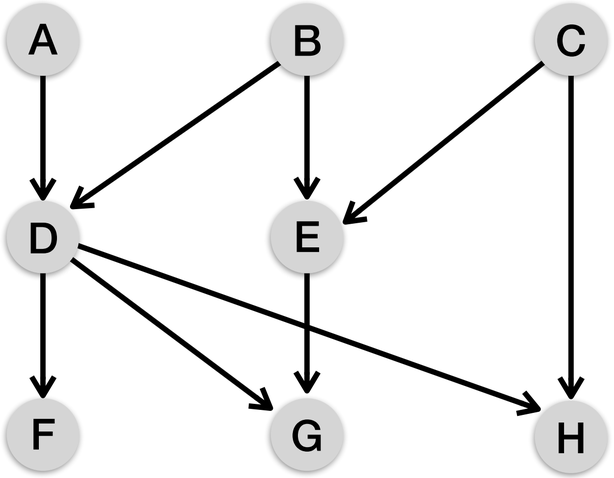
Algorithms and Data Structures Recap

Topological sort/ordering

- A *topological ordering* of a directed graph is a linear ordering of its vertices such that for every directed edge (u, v) from vertex u to vertex v, u comes before v in the ordering.
- A *directed acyclic graph* (DAG) is a directed graph without directed cycles.
- A topological ordering is only possible if and only if the graph is a DAG.

Example

$$G = (V, E)$$



$$A, B, C, D, E, F, G, H$$

(*) note that G has many topological orderings. Another one is C, A, B, E, D, F, G, H.