1. What is the probability that after you and your friend have played one turn each, your meeples are on the same position, and that position is indexed with an even number?

Let $A :=$ "meeples on same position and position even".

The dice are independant, hence

$$
\begin{aligned}
Pr[A] = \; & Pr[\text{"both on pos 2"}] + \\
& Pr[\text{"both on pos 4"}] + \\
& Pr[\text{"both on pos 6"}] \\
= \; & \tfrac{1}{6} \cdot p_2 + \tfrac{1}{6} \cdot p_4 + \tfrac{1}{6} \cdot p_6 = \tfrac{1}{6} (p_2 + p_4 + p_s).
\end{aligned}
$$

2. Conditioned on your meeple landing on position 7 after your second die roll, what is the probability that your meeple landed on position 3 after your first die roll?

Let $A :=$ "meeple on position 3 after 1^{st} roll" and
$B :=$ "meeple on position 7 after 2^{nd} roll".

$\Pr[A \mid B] = ?$

Satz 2.15. *(Satz von Bayes)* Die Ereignisse $A_1, \ldots, A_n$ seien paarweise disjunkt. Ferner sei $B \subseteq A_1 \cup \cdots \cup A_n$ ein Ereignis mit $\Pr[B] > 0$. Dann gilt für ein beliebiges $i = 1, \ldots, n$

$$\Pr[A_i|B] = \frac{\Pr[A_i \cap B]}{\Pr[B]} = \frac{\Pr[B|A_i] \cdot \Pr[A_i]}{\sum_{j=1}^{n} \Pr[B|A_j] \cdot \Pr[A_j]}.$$

Let $\Omega = \{1, \ldots, 6\}^2$ s.t. $\omega = (i,j) \in \Omega$ can be understood as first throw shows i and second throw shows j.

Let $A_i :=$ "meeple on position i after 1^{st} roll", i.e.

$A_i = \{(i,j) \mid j \in \{1, \ldots, 6\}\}$.

It's easy to see that these events are pairwise disjoint.

Let $B :=$ "meeple on position 7 after 2^{nd} roll", i.e.

$B = \{(i,j) \mid i+j = 7,\ 1 \leq i,j \leq 6\}$

$= \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$.

It's easy to see that $B \subseteq \bigcup_{i=1}^{6} A_i$ and $\Pr[B] > 0$.

We use Bayes' theorem for A_3:

$$\Pr[A_3 \mid B] = \frac{\Pr[A_3 \cap B]}{\Pr[B]} .$$

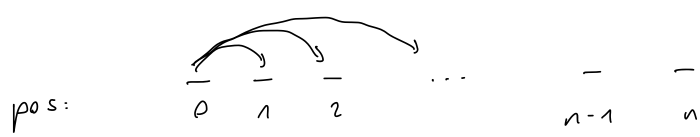
3. Independently of your friend's meeple's behaviour, what is the expected number of die rolls it will take you to reach position n?

[dynamic programming recap on slides]

Let X be a random variable that counts the number of rolls to go from position 0 to n.

We want to compute $\mathbb{E}[X]$.

Ideally, we would like to split $\mathbb{E}[X]$ into suitable subproblems so we can find a recurrence formula.



We define

$$f(x) := \mathbb{E}[\text{"\# rolls to go from } 0 \text{ to } n\text{"}], \quad 0 \leq x \leq n.$$

Trivially, we have $f(n) = 0$ as it takes 0 rolls to get from n to n. (Base case)

Consider $f(n-1)$: ($\mathbb{E}[$"from $n-1$ to n"$]$)

$$\underline{n-1} \xrightarrow{1,2,3,4,5,6} \underline{n}$$

(Recall: If on position x we roll i s.t. $x+i \geq n$ we land on n)

Lemma 2.29. Ist X eine Zufallsvariable, so gilt:

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr[\omega].$$

Let Ω be the set of possible roll sequences to get from $n-1$ to n, i.e. $\Omega = \{(1), (2), (3), \ldots, (6)\}$.

Then

$$\sum_{\omega \in \Omega} \underbrace{X(\omega)}_{\text{only one roll.}} \underbrace{\Pr[\omega]}_{\text{just the probability of rolling } i.} = \sum_{i=1}^{6} 1 \cdot p_i = 1.$$

(this makes sense of course)

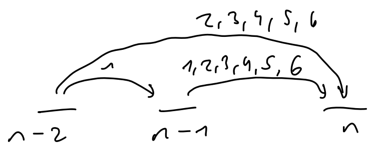
So far we have seen

$$f(n) = \mathbb{E}[\text{"\# rolls to go from } n \text{ to } n\text{"}] = 0,$$

$$f(n-1) = \mathbb{E}[\text{"\# rolls to go from } n \text{ to } n\text{"}] = 1.$$

Consider $f(n-2)$:

$(\mathbb{E}[\text{"from } n-1 \text{ to } n\text{"}])$



(Recall: if on position x we roll i s.t. $x + i \geq n$ we land on n)

$$\Omega = \{ (1,1), (1,2), (1,3), \dots, (1,6), \quad \} \text{ 2 rolls}$$
$$(2), (3), (4), \dots, (6) \} \quad \} \text{ 1 roll}$$

$$\sum_{\omega \in \Omega} X(\omega) \Pr(\omega) = X((1,1)) \cdot \Pr[(1,1)] + X((1,2)) \cdot \Pr([1,2)] + \dots$$
$$X((2)) \cdot \Pr((2)) + \dots + X((6)) \Pr[(6)]$$

$$= 2p_1^2 + 2p_1p_2 + 2p_1p_3 + 2p_1p_4 + 2p_1p_5 + 2p_1p_6 +$$
$$1p_2 + 1p_3 + 1p_4 + 1p_5 + 1p_6$$

We are now looking for a recurrence formula.

So far we know $f(n-1)$ and $f(n)$.

Notice how for the sequences with 2 rolls we go from $n-2$ to $n-1$ first.

$$= 2p_1 \overbrace{(p_1 + p_2 + p_3 + p_4 + p_5 + p_6)}^{(*) \text{ looks like } f(n-1)\dots}$$
$$+ p_2 + p_3 + p_4 + p_5 + p_6$$

The factorization $(*)$ seems to make sense, as we first go from $n-2$ to $n-1$ with a roll that shows 1, and then from $n-1$ to n.

<u>First idea:</u> $f(n-2) \overset{!}{\approx} \sum_{i=1}^{6} p_i f(n-2+i)$

$$f(n-2) \overset{!}{\approx} p_1 f(n-1) + p_2 f(n) + p_3 f(n+1) + \dots + p_6 f(n)$$

We can't go over n, so let's write

$f(x) := f(n) \quad x > n.$

$$f(n-2) \overset{!}{\approx} p_1 f(n-1) + p_2 f(n) + p_3 f(n) + \dots + p_6 f(n)$$

$$= p_1 + p_2 \cdot 0 + \dots + p_6 \cdot f(n) = p_1 \quad \lightning$$

Remember $f(x)$ is defined as some kind of expected value <u>not</u> as some probability.

So if $f(n-1)$ is the expected # of rolls to go from $n-1$ to n, and we go from $n-2$ to $n-1$ using <u>a single roll</u> we have to <u>account for that roll</u>.

Second idea: $f(n-2) \overset{!}{\approx} \sum_{i=1}^{6} p_i \left(1 + f(n-2+i)\right)$

one roll.

$$f(n-2) = p_1(1+f(n-1)) + p_2(1+f(n)) + p_3(1+f(n+1)) + \dots + p_6(1+f(n+5))$$

We can't go over n, so we define

$$f(x) := f(n) \qquad x > n.$$

$$f(n-2) = p_1(1+f(n-1)) + p_2(1+f(n)) + p_3(1+f(n)) + \dots + p_6(1+f(n))$$

$$= p_1(1+1) + p_2(1+0) + \dots + p_6(1+0)$$

$$= 2p_1 + p_2 + \dots + p_6.$$

This matches our previous computation.

we derive the following recurrence:

$$f(x) = \sum_{i=1}^{6} p_i \left(1 + f(x+i)\right) = 1 + \sum_{i=1}^{6} p_i f(x+i)$$

recall $\sum_{i=1}^{6} p_i = 1$.

for $0 \leq x < n$ and $f(y) = f(n)$, $y > n$.