Math Basics for Asymptotic Analysis

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1. Continuous Functions

- Functions that have no breaks, jumps, or holes.
- You can draw them without lifting your pen.

2. Differentiation

What it is: Differentiation is the process of finding the derivative of a function, which represents the rate of change of the function with respect to a variable.

Common Differentiation Rules:

- **Power Rule**: $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$
- **Sum Rule**: $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$
- **Product Rule**: $\frac{d}{dx}(f(x) \cdot g(x)) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
- **Quotient Rule**: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$
- **Chain Rule**: $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$

3. Limits

- Describes the value a function approaches as the input gets closer to a certain point.
- **Tending to infinity** is especially important in algorithms for understanding asymptotic behavior, e.g., how a function behaves as $n \to \infty$.

4. Limit Rules

Assuming the limits exists, we have:

- $\lim_{n\to\infty}(f(n) \pm g(n)) = \lim_{n\to\infty} f(n) \pm \lim_{n\to\infty} g(n)$.
- $\lim_{n\to\infty}(c_1 \cdot f(n) + c_2) = c_1 \cdot (\lim_{n\to\infty} f(n)) + c_2$, for some constants c_1, c_2
- $\lim_{n\to\infty} f(n) \cdot g(n) = \lim_{n\to\infty} f(n) \cdot \lim_{n\to\infty} g(n)$.
- $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{\lim_{n\to\infty} f(n)}{\lim_{n\to\infty} g(n)}$, if $\lim_{n\to\infty} g(n) \neq 0$.

5. Logarithms

- The inverse of exponentiation: $\log_b(x)$ answers the question: "To what power must b be raised to get x?". For example, to what power must 2 be raised to get 8? Answer: $\log_2(8) = 3$ times such that $2^{\log_2(8)} = 2^3 = 8$.
- The natural logarithm denoted by $\ln$ has base e.

6. Logarithm Rules

- $\log_a(x \cdot y) = \log_a x + \log_a y$.
- $\log_a \frac{x}{y} = \log_a x - \log_a y$.
- $\log_a x^n = n \cdot \log_a x$.
- $\log_a b = \frac{\log_c b}{\log_c a}$.
- $\log_a b = \frac{1}{\log_b a}$.

7. Exponents

- Express repeated multiplication: $a^n = a \cdot a \cdot \ldots \cdot a$ (n times)

8. Exponent Rules

- $x^m \cdot x^n = x^{m+n}$.
- $\frac{x^m}{x^n} = x^{m-n}$.
- $(x^m)^n = x^{m \cdot n}$.
- $(x \cdot y)^n = x^n \cdot y^n$ and $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$.
- $x^0 = 1$.
- $x^{-n} = \frac{1}{x^n}$.
- $x^{\frac{m}{n}} = \sqrt[n]{x^m}$.

9. Big Sum and Product Notation

- **Big Sum (Sigma Notation)**: $\sum_{i=m}^{n} a_i$ represents the sum of the terms a_i from index $i = m$ to $i = n$. For example, $\sum_{i=1}^{4} i = 1 + 2 + 3 + 4 = 10$.
- **Big Product (Pi Notation)**: $\prod_{i=m}^{n} a_i$ represents the product of the terms a_i from index $i = m$ to $i = n$. For example, $\prod_{i=1}^{4} i = 1 \cdot 2 \cdot 3 \cdot 4 = 24$.