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Algorithms & Data Structures — Exercise sheet 1 — HS 23

The solutions for this sheet are submitted at the beginning of the exercise class on 2 October 2023.

Exercises that are marked by * are challenge exercises. They do not count towards bonus points.

You can use results from previous parts without solving those parts.

Exercise 1.1 *Guess the formula* **(1 point)**.

Consider the recursive formula defined by $a_1 = 2$ and $a_{n+1} = 3a_n - 2$ for $n > 1$. Find a simple closed formula for a_n and prove that a_n follows it using mathematical induction.

Hint: *Write out the first few terms. How fast does the sequence grow?*

Exercise 1.2 *Sum of Cubes* **(1 point)**.

Prove by mathematical induction that for every positive integer n,

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

Exercise 1.3 *Sums of powers of integers.*

In this exercise, we fix an integer $k \in \mathbb{N}_0$.

(a) Show that, for all $n \in \mathbb{N}_0$, we have $\sum_{i=1}^{n} i^k \leq n^{k+1}$.

(b) Show that for all $n \in \mathbb{N}_0$, we have $\sum_{i=1}^{n} i^k \geq \frac{1}{2^{k+1}} \cdot n^{k+1}$.

Hint: *Consider the second half of the sum, i.e., $\sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^k$. How many terms are there in this sum? How small can they be?*

Together, these two inequalities show that $C_1 \cdot n^{k+1} \leq \sum_{i=1}^{n} i^k \leq C_2 \cdot n^{k+1}$, where $C_1 = \frac{1}{2^{k+1}}$ and $C_2 = 1$ are two constants independent of n. Hence, when n is large, $\sum_{i=1}^{n} i^k$ behaves "almost like n^{k+1}" up to a constant factor.

Exercise 1.4 *Asymptotic growth* **(1 point)**.

Recall the concept of asymptotic growth that we introduced in Exercise sheet 0: If $f, g : \mathbb{N} \to \mathbb{R}^+$ are two functions, then:

- We say that f *grows asymptotically slower than* g if $\lim_{m\to\infty} \frac{f(m)}{g(m)} = 0$. If this is the case, we also say that g *grows asymptotically faster than* f.

Prove or disprove each of the following statements.

(a) $f(m) = 10m^3 - m^2$ grows asymptotically slower than $g(m) = 100m^3$.

(b) $f(m) = 100 \cdot m^2 \log(m) + 10 \cdot m^3$ grows asymptotically slower than $g(m) = 5 \cdot m^3 \log(m)$.

Hint: $\log(m)$ *grows asymptotically slower than* m.

(c) $f(m) = \log(m)$ grows asymptotically slower than $g(m) = \log(m^4)$.

(d) $f(m) = 2^{(0.9m^2+m)}$ grows asymptotically slower than $g(m) = 2^{(m^2)}$.

(e) If f grows asymptotically slower than g, and g grows asymptotically slower than h, then f grows asymptotically slower than h.

Hint: *For any* $a, b : \mathbb{N} \to \mathbb{R}^+$, *if* $\lim_{m\to\infty} a(m) = A$ *and* $\lim_{m\to\infty} b(m) = B$, *then* $\lim_{m\to\infty} a(m)b(m) = AB$.

(f) If f grows asymptotically slower than g, and $h : \mathbb{N} \to \mathbb{N}$ grows asymptotically faster than 1, then f grows asymptotically slower than $g(h(m))$.

Exercise 1.5 *Proving Inequalities.*

(a) By induction, prove the inequality

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}, \quad n \geq 1.$$

(b)* Replace $3n + 1$ by $3n$ on the right side, and try to prove the new inequality by induction. This inequality is even weaker, hence it must be true. However, the induction proof fails. Try to explain to yourself how is this possible?

bonus points (out of 3).

Exercise 1.1 *Guess the formula* **(1 point)**.

Consider the recursive formula defined by $a_1 = 2$ and $a_{n+1} = 3a_n - 2$ for $n > 1$. Find a simple closed formula for a_n and prove that a_n follows it using mathematical induction.

part 1

Hint: *Write out the first few terms. How fast does the sequence grow?*

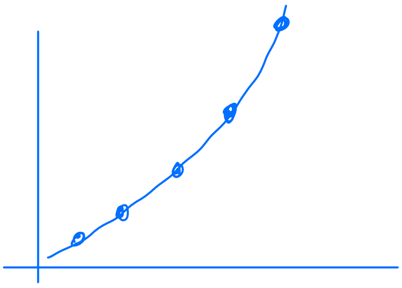
part 2

Hints are very useful in many cases.

$a_1 = 2, \quad a_2 = 3 \cdot 2 - 2 = 4, \quad \dots$

$2, 4, 10, 28, 82.$ } part 1.

How fast does it grow?



doesn't seem linear.

polynom? exponential?

seems to be $\approx \cdot 3$?

$f: \mathbb{N} \to \mathbb{R}$ such that $f(n) = a_n$.

a_n seems to be even?

$\Longrightarrow \quad a_n = 3^{n-1} + 1$

- **Base Case.**
  For $n = 1$ we have
  $$a_1 = 2 = 3^{1-1} + 1,$$
  so it is true for $n = 1$.
- **Induction Hypothesis.**
  We now assume that it is true for $n = k$, i.e., $a_k = 3^{k-1} + 1$.
- **Induction Step.**
  We want to prove that it is also true for $n = k + 1$. Using the induction hypothesis we get

$$a_{k+1} = 3a_k - 2 = 3 \cdot (3^{k-1} + 1) - 2 = 3 \cdot 3^{k-1} + 3 - 2 = 3^k + 1.$$

Hence, it is true for $n = k + 1$.

By the principle of mathematical induction, we conclude that $a_n = 3^{n-1} + 1$ is true for any positive integer n.

Exercise 1.2 *Sum of Cubes* **(1 point).**

Prove by mathematical induction that for every positive integer n,

$$\underbrace{1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}}_{\text{very similar to last week's exercise.}}.$$

- **Base Case.**
  Let $n = 1$. Then,

$$1^3 = 1 = \frac{1^2 \cdot (1+1)^2}{4},$$

  so the property holds for $n = 1$.

- **Induction Hypothesis.**
  Assume that the property holds for some positive integer k, that is,

$$1^3 + 2^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}.$$

- **Induction Step.**
  We must show that the property also holds for $k+1$. Let us add $(k+1)^3$ to both sides of the induction hypothesis. We get

Tip #1: sometimes, using $\sum$ ($\prod$) - notation can be easier than writing out the terms. (+ less error prone)

e.g. $1^3 + 2^3 + \ldots + k^3 = \sum_{i=1}^{k} i^3$.

$$\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3 \overset{I.H.}{=} \frac{k^2(k+1)^2}{4} + (k+1)^2$$

$$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

now we could just expand everything, but... $\Rightarrow$ error prone + difficult!

$$= \frac{(k+1)^2(k^2 + 4(k+1))}{4}$$

same argument! look closely + think it through!

$$= \frac{(k+1)^2(k^2 + 4k + 4)}{4}$$

being sufficient at binomial expansion really helps!

$$= \frac{(k+1)^2 (k+2)^2}{4} = \frac{(k+1)^2 ((k+1)+1)^2}{4}.$$

By the principle of mathematical induction, $1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ is true for any positive integer n.

Exercise 1.4 *Asymptotic growth* **(1 point)**.

Recall the concept of asymptotic growth that we introduced in Exercise sheet 0: If $f, g : \mathbb{N} \to \mathbb{R}^+$ are two functions, then:

- We say that f *grows asymptotically slower than* g if $\lim_{m\to\infty} \frac{f(m)}{g(m)} = 0$. If this is the case, we also say that g *grows asymptotically faster than* f.

Prove or disprove each of the following statements.

(a) $f(m) = 10m^3 - m^2$ grows asymptotically slower than $g(m) = 100m^3$.

Solution:

False, since

$$\begin{aligned}\lim_{m\to\infty} \frac{f(m)}{g(m)} &= \lim_{m\to\infty} \frac{10m^3 - m^2}{100m^3} \\ &= \lim_{m\to\infty} \frac{1}{10} - \frac{1}{100m} = \frac{1}{10} + 0 > 0.\end{aligned}$$

(b) $f(m) = 100 \cdot m^2 \log(m) + 10 \cdot m^3$ grows asymptotically slower than $g(m) = 5 \cdot m^3 \log(m)$.

Hint: $\log(m)$ *grows asymptotically slower than* m.

Solution:

True, since

$$\begin{aligned}\lim_{m\to\infty} \frac{f(m)}{g(m)} &= \lim_{m\to\infty} \frac{100 \cdot m^2 \log(m) + 10 \cdot m^3}{5 \cdot m^3 \log(m)} \\ &= \lim_{m\to\infty} \frac{20}{m} + \frac{2}{\log(m)} = 0 + 0.\end{aligned}$$

(c) $f(m) = \log(m)$ grows asymptotically slower than $g(m) = \log(m^4)$.

Solution:

False, since

$$\begin{aligned}\lim_{m\to\infty} \frac{f(m)}{g(m)} &= \lim_{m\to\infty} \frac{\log(m)}{\log(m^4)} \\ &= \lim_{m\to\infty} \frac{\log(m)}{4\log(m)} = \frac{1}{4} > 0.\end{aligned}$$

(d) $f(m) = 2^{(0.9m^2+m)}$ grows asymptotically slower than $g(m) = 2^{(m^2)}$.

Solution:

True, since

$$\begin{aligned}\lim_{m\to\infty} \frac{f(m)}{g(m)} &= \lim_{m\to\infty} \frac{2^{(0.9m^2+m)}}{2^{(m^2)}} \\ &= \lim_{m\to\infty} 2^{m-0.1m^2} = 0,\end{aligned}$$

as $m - 0.1m^2 \to -\infty$ as $m \to \infty$.

(e) If f grows asymptotically slower than g, and g grows asymptotically slower than h, then f grows asymptotically slower than h.

Hint: *For any* $a, b : \mathbb{N} \to \mathbb{R}^+$, *if* $\lim_{m\to\infty} a(m) = A$ *and* $\lim_{m\to\infty} b(m) = B$, *then* $\lim_{m\to\infty} a(m)b(m) = AB$.

$$\lim_{m\to\infty} \frac{f(m)}{h(m)} = \lim_{m\to\infty} \frac{f(m)}{h(m)} \cdot 1$$

trick used in maths!

$$= \lim_{m\to\infty} \frac{f(m)}{h(m)} \frac{g(m)}{g(m)}$$

$$= \lim_{m\to\infty} \frac{f(m)}{g(m)} \frac{g(m)}{h(m)}$$

$$= \underbrace{\lim_{m\to\infty} \frac{f(m)}{g(m)} \cdot \lim_{m\to\infty} \frac{g(m)}{h(m)}}_{} = 0.$$

how do I see this more easily?
translation from words → math and vice versa can be hard

$$\boxed{\begin{array}{l}\text{Trick \# 2: summarise the information you get from the description of the problem.} \\ \text{e.g. } \lim_{m\to\infty} \frac{f(m)}{g(m)} = 0, \quad \lim \frac{g(m)}{h(m)} = 0 \text{ and hint.}\end{array}}$$

(f) If f grows asymptotically slower than g, and $h : \mathbb{N} \to \mathbb{N}$ grows asymptotically faster than 1, then f grows asymptotically slower than $g(h(m))$.

Solution:

False, consider $f(m) = 1/m^2$, $g(m) = 1/m$ and $h(m) = m^3$. They satisfy the conditions, but f does not grow slower than $g(h(m)) = 1/m^3$.

I'm sure this might be confusing to some.
My suggestion, focus on what you have and what you need to prove:

through the description

We have:

- $\lim_{m \to \infty} \frac{f(m)}{g(m)} = 0$
- $h : \mathbb{N} \to \mathbb{N}$, $\lim_{m \to \infty} \frac{1}{h(m)} = 0$

I need to show:

- $\lim_{m \to \infty} \frac{f(m)}{g(h(m))} = 0$

What we would need to prove formally:

$\forall$ functions f, g, h that fulfill properties above: it is true that $\lim_{m \to \infty} \frac{f(m)}{g(h(m))} = 0$

What we would need to disprove formally

$\neg$ ($\forall$ functions f, g, h that fulfill properties above: it is true that $\lim_{m \to \infty} \frac{f(m)}{g(h(m))} = 0$)

Disc. Math

$\iff$ $\exists$ functions f, g, h —"— $\lim_{m \to \infty} \frac{f(m)}{g(h(m))} \neq 0$

Many students used this <u>wrong</u> proof:

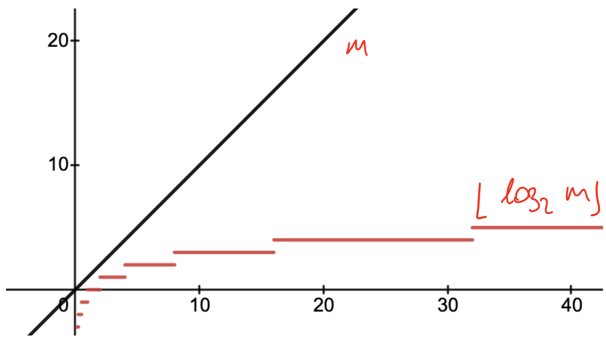
- we know: $\lim\limits_{m\to\infty} \frac{f(m)}{g(m)} = 0$
- $\lim\limits_{m\to\infty} \frac{1}{h(m)} = 0 \implies \lim\limits_{m\to\infty} h(m) = \infty$.
- $\lim\limits_{m\to\infty} \frac{f(m)}{g(h(m))} \stackrel{!}{=} 0$

Yes, because as $m \to \infty$ $h(m) \to \infty$, and since $h: \mathbb{N} \to \mathbb{N}$, $g(h(m))$ grows at least as fast as $g(m)$ which doesn't grow slower than $f(m)$.

This is wrong! But why?

<u>Case 1:</u> f, g are increasing $(\uparrow)$

It is <u>wrong</u> to assume that $h(m) = m$ is the "slowest" growing function from $\mathbb{N} \to \mathbb{N}$. Consider $\lfloor \log_2 m \rfloor$. E.g. $f = m$, $g = m^2$



then:

$$\lim_{m\to\infty} \frac{f(m)}{g(h(m))}$$

$$= \lim_{m\to\infty} \frac{m}{\lfloor \log_2 m \rfloor^2}$$

L'Hôpital

Now since $\lim\limits_{m\to\infty} \frac{m}{(\log_2 m)^2} = \infty$

and $\frac{n}{(\log_2 n)^2} \leq \frac{n}{\lfloor \log_2 n \rfloor^2} \Rightarrow \lim_{n\to\infty} \frac{n}{\lfloor \log_2 n \rfloor^2} = \infty$.

We found a counter example!

Case 2: f, g are decreasing (↓)

Counter examples where f, g are decreasing might not come as intuitive as Case 1, but as long as the functions fulfil the needed properties they are correct. For example:

$f(n) = 1/n^2$, $g(n) = 1/n$, $h(n) = n^3$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n} = 0$$

$\Rightarrow$ f grows slower than g.

But

$$\lim_{n\to\infty} \frac{f(n)}{g(h(n))} = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n^3}} = \lim_{n\to\infty} n = \infty.$$

$\lightning$

How do I find such a solution?

For Case 1 it's just about finding a function $h : \mathbb{N} \to \mathbb{N}$ that grows slower then m.

For <u>case 2</u>, the following trial and error approach could lead to a solution. Let's say we try $f = \frac{1}{m}$. Now we choose $g = m$ (because it's simple) and $h = m$ (—"—).

$$\lim_{m\to\infty} \frac{f(m)}{g(m)} = \lim_{m\to\infty} \frac{\frac{1}{m}}{m} = \lim_{m\to\infty} \frac{1}{m^2} = 0 \quad \checkmark$$

Now

$$\lim_{m\to\infty} \frac{f(m)}{g(h(m))} = \lim_{m\to\infty} \frac{\frac{1}{m}}{m} = \ldots = 0$$

If h grows any faster than m, the limit stays 0. We try to modify g. Let $g = \frac{1}{m}$, but then f has to change too, else f and g grow equally as fast $\Rightarrow f = \frac{1}{m^2}$.

$$\lim_{m\to\infty} \frac{f(m)}{g(h(m))} = \lim_{m\to\infty} \frac{\frac{1}{m^2}}{\frac{1}{m}} = \lim_{m\to\infty} \frac{1}{m} = 0$$

We notice h has to be faster than m^2 such that the limit goes to ∞ $\Rightarrow h = m^3$.

Exercise 1.3 *Sums of powers of integers.*

In this exercise, we fix an integer $k \in \mathbb{N}_0$.

(a) Show that, for all $n \in \mathbb{N}_0$, we have $\sum_{i=1}^{n} i^k \leq n^{k+1}$.

Solution:

As all terms in the sum are at most n^k, we have:

in my guide.

$$\sum_{i=1}^{n} i^k \leq \sum_{i=1}^{n} n^k = n \cdot n^k = n^{k+1}.$$

(b) Show that for all $n \in \mathbb{N}_0$, we have $\sum_{i=1}^{n} i^k \geq \frac{1}{2^{k+1}} \cdot n^{k+1}$.

Hint: *Consider the second half of the sum, i.e.,* $\sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^k$. *How many terms are there in this sum? How small can they be?*

$\lceil \cdot \rceil$ ceil! (what if $\frac{n}{2} \notin \mathbb{N}$..!)

$$\sum_{i=1}^{n} i^k = 1^k + 2^k + 3^k + \ldots + \lceil \tfrac{n}{2} \rceil^k + \left(\lceil \tfrac{n}{2} \rceil + 1\right)^k + \ldots + n^k.$$

$$\sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^k = \cancel{1^k + 2^k + 3^k + \ldots +} \lceil \tfrac{n}{2} \rceil^k + \left(\lceil \tfrac{n}{2} \rceil + 1\right)^k + \ldots + n^k.$$

$>0 \quad >0 \quad >0 \quad \ldots$

$$\Longrightarrow \sum_{i=1}^{n} i^k \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^k \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \left(\frac{n}{2}\right)^k$$

now solutions suggest:

$$\sum_{i=1}^{n} i^k \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^k \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \left(\frac{n}{2}\right)^k \geq \left(n - \lceil \tfrac{n}{2} \rceil + 1\right) \cdot \left(\frac{n}{2}\right)^k$$

by def of $\lceil \cdot \rceil$, we have $\lceil \frac{n}{2} \rceil - 1 \leq \frac{n}{2} \qquad | \cdot (-1)$

$$\Longleftrightarrow -\lceil \tfrac{n}{2} \rceil + 1 \geq -\frac{n}{2} \qquad | + n$$

$$\Longleftrightarrow n - \lceil \tfrac{n}{2} \rceil + 1 \geq \frac{n}{2}$$

$$\Longrightarrow \quad \sum_{i=1}^{n} i^k \geq \frac{n}{2} \cdot \left(\frac{n}{2}\right)^k = \frac{1}{2^{k+1}} \cdot n^{k+1}$$

I suggest:

$n = \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor$, thus $n - \lceil \frac{n}{2} \rceil + 1 = \lfloor \frac{n}{2} \rfloor + 1$, hence

$$\sum_{i=1}^{n} i^k \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^k \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \left(\frac{n}{2}\right)^k = \left(\lfloor \frac{n}{2} \rfloor + 1\right)\left(\frac{n}{2}\right)^k$$

It is easy to see that $\lfloor \frac{n}{2} \rfloor + 1 \geq \frac{n}{2}$, thus

$$\sum_{i=1}^{n} i^k \geq \frac{n}{2} \left(\frac{n}{2}\right)^{k+1} = \frac{1}{2^{k+1}} \, n^{k+1} \, .$$

Together, these two inequalities show that $C_1 \cdot n^{k+1} \leq \sum_{i=1}^{n} i^k \leq C_2 \cdot n^{k+1}$, where $C_1 = \frac{1}{2^{k+1}}$ and $C_2 = 1$ are two constants independent of n. Hence, when n is large, $\sum_{i=1}^{n} i^k$ behaves "almost like n^{k+1}" up to a constant factor.

Exercise 1.5 *Proving Inequalities.*

(a) By induction, prove the inequality

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}, \qquad n \geq 1.$$

refer to official solutions.

(b)* Replace $3n+1$ by $3n$ on the right side, and try to prove the new inequality by induction. This inequality is even weaker, hence it must be true. However, the induction proof fails. Try to explain to yourself how is this possible?

In what sense "weaker"? Look here:

(https://math.stackexchange.com/questions/2543452/what-are-the-strong-and-weak-in-mathematics)

Solution:

(b) Sometimes it is easier to prove more than less. This simple approach does not work for the weaker inequality as we are using a weaker (and insufficiently so!) induction hypothesis in each step.

Not necessary, for completeness.

IA: $n = 1$

$$\prod_{i=1}^{n} \frac{2i-1}{2i} = \frac{1}{2} \leq \frac{1}{\sqrt{3}}$$

IH: Assume it is true for some n: $\displaystyle\prod_{i=1}^{n} \frac{2i-1}{2i} \leq \frac{1}{\sqrt{3n}}$

IS: $n \rightsquigarrow n+1$

$$\prod_{i=1}^{n+1} \frac{2i-1}{2i} = \frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2n-1}{2n} \cdot \frac{2n+1}{2n+2}$$

$$\leq \frac{1}{\sqrt{3n}} \cdot \frac{2n+1}{2n+2}$$

we now show that $\forall n \in \mathbb{N}_{>0}$

$$\frac{1}{\sqrt{3n}} \frac{2n+1}{2n+2} \leq \frac{1}{\sqrt{3n+3}} = \frac{1}{\sqrt{3(n+1)}} \quad \Big|^2$$

$$\frac{(2n+1)^2}{(3n)(2n+2)^2} \leq \frac{1}{3n+3}$$

$$\frac{(3n)(2n+2)^2}{(2n+1)^2} \geq 3n+3$$

$$(3n)(2n+2)^2 \geq (3n+3)(2n+1)^2$$

$$12n^3 + 24n^2 + 12n \geq 12n^3 + 24n^2 + 15n + 3$$

$$0 \geq 3n + 3$$

$$n \leq -1$$

which is a contradiction because we defined n to be greater than or equal to 1.