Exercise 12.4 *TST and MST* **(1 point)**.

Let $G = (V, E)$ be a connected, weighted graph, with weights $w : E \to \mathbb{R}_{\geq 0}$. A *travelling salesperson tour* (TST) in G is a closed walk which visits each vertex $v \in V$ at least once. We write $\mathrm{tst}(G)$ for the length of a shortest TST in G, that is:

$$\mathrm{tst}(G) = \min_{\substack{P=(v_1,\ldots,v_\ell) \\ \text{is a TST in } G}} w(P), \quad \text{where } w(P) := \sum_{i=1}^{\ell-1} w(\{v_i, v_{i+1}\}).$$

(a) Let $M \subseteq E$ be the edges of a minimum spanning tree of G, with weight $w(M) := \sum_{e \in M} w(e)$. Prove that $w(M) \leq \mathrm{tst}(G)$.

Let P be a TST in G, and let $E(P)$ be the set of edges traversed by P. Then $E(P)$ spans V. Therefore, the graph $G' = (V, E(P))$ is connected, and thus it has a spanning tree T, whose weight is at most $w(P)$. But T is also a spanning tree for G, and so $w(M) \leq w(T) \leq w(P)$.

Definition: A subgraph is spanning when it includes all vertices of the given graph.

Let P be a TST in G and let $E(P)$ be the set of edges traversed by P. Since P visits all vertices in G at least once, the subgraph $G' = (V, E(P))$ is spanning. Since G and G' have the same vertices and we the set of edges in G' is the set of edges traversed by P it is easy to see that P is also a TST in G'. Therefore, G' is also connected and thus it has a spanning tree T, whose weight is at most $w(P)$. But T is also a spanning tree for

G, and so $w(M) \leq w(T) \leq w(P)$

(b) Let $H = (V, M_{\text{double}})$ be the multigraph with vertex set V, and edge set M_{double} containing two copies of each edge $e \in M$. Prove that H has a Eulerian tour of length $2 \cdot w(M)$.

Hint: *See Exercise 10.1. What can you say about the degree of a vertex in H?*

Solution:

As we have doubled all edges in M to obtain M_{double}, each vertex $v \in V$ has even degree (in H). But this implies that H has a Eulerian tour. (To see this, we can use the construction of Exercise 10.1, which shows H has a Eulerian tour if and only if the (simple) graph H' obtained by subdivision of the edges of H has a Eulerian tour. The vertices of that graph all have even degree, and for simple graphs we know that this is equivalent to having a Eulerian tour). The length of a Eulerian tour in H is just $\sum_{e \in M_{\text{double}}} w(e) = 2 \sum_{e \in M} w(e) = 2 \cdot w(M)$.

(c) Describe an algorithm which outputs a TST in G of length at most $2 \cdot \text{mst}(G)$, where $\text{mst}(G)$ is the length of a minimum spanning tree of G. The runtime of your algorithm should be at most $O(|E| \log |E|)$. Prove that your algorithm is correct and achieves the desired runtime.

Hint: *For a connected graph with n vertices and m edges, you may use the fact that there exists an algorithm to find a minimum spanning tree in time $O(m \log m)$, and a Eulerian tour (if one exists) in time $O(m)$.*

(1) find MST T in G with edges $M \subseteq E$

(2) construct multigraph $H = (V, M_{\text{double}})$

(3) construct simple graph H' from H (ex. 10.1.)

(4) find Eulerian tour P in H'.

(5) output P.

Correctness: G is connected, therefore we can find a MST in (1). Correctness of (3) follows from 10.1. Since P from (4) is a Eulerian tour in H', by exercise 10.1. we know there is also a Eulerian tour P' in H.
Since every vertex and edge we used in P' is also in G by construction of H, P' is a closed walk that visits every vertex in G at least once. Therefore P' is a TST in G with weight (at most) $2\,w(M) = 2\,\mathrm{mst}(G)$.

Runtime:

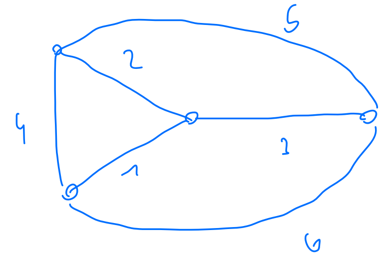
(1) $O(m \log m)$ (hint)
(2) + (3) $O(n+m)$ (ex. 10.1)
(4) + (5) $O(m)$

$\Longrightarrow$ $O(m \log m)$

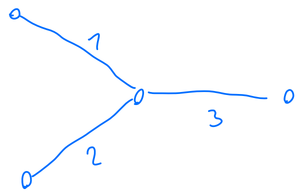
where we used that $2m \geq n \iff m \geq \frac{1}{2} n$.

Example

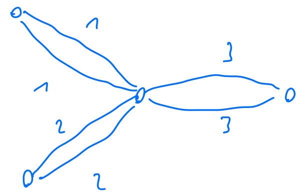
G :



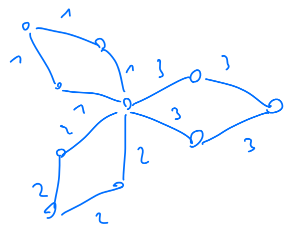
MST $\Longrightarrow$



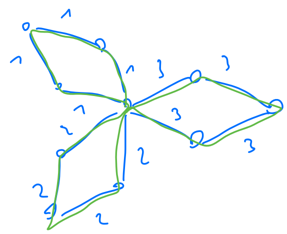
H $\Longrightarrow$



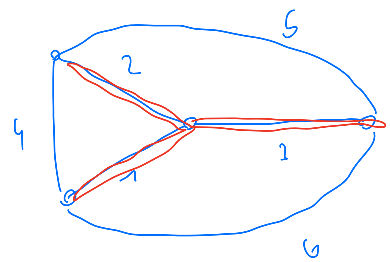
H' $\Longrightarrow$



Eulerian tour P $\Longrightarrow$



walk P' in G $\Longrightarrow$



$w(P') = 2\,mst(G)$, where P' is TST in G.