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25 September 2023

Algorithms & Data Structures — Exercise sheet 0 — HS 23

The solutions for this sheet do not have to be submitted. The sheet will be solved in the first exercise session on 25.09.2023.

Exercises that are marked by * are challenge exercises.

Exercise 0.1 *Induction.*

a) Prove by mathematical induction that for any positive integer n,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

b) **(This subtask is from August 2019 exam).** Let $T : \mathbb{N} \to \mathbb{R}$ be a function that satisfies the following two conditions:

$$T(n) \geq 4 \cdot T(\tfrac{n}{2}) + 3n \qquad \text{whenever } n \text{ is divisible by 2;}$$
$$T(1) = 4.$$

Prove by mathematical induction that

$$T(n) \geq 6n^2 - 2n$$

holds whenever n is a power of 2, i.e., $n = 2^k$ with $k \in \mathbb{N}_0$.

Asymptotic Growth

When we estimate the number of elementary operations executed by algorithms, it is often useful to ignore smaller order terms, and instead focus on the asymptotic growth defined below. We denote by $\mathbb{R}^+$ the set of all (strictly) positive real numbers and by $\mathbb{R}_0^+$ the set of nonnegative real numbers.

Definition 1. Let $f, g : \mathbb{N} \to \mathbb{R}^+$ be two functions. We say that f *grows asymptotically faster than* g if $\lim\limits_{n\to\infty} \frac{g(n)}{f(n)} = 0$.

This definition is also valid for functions defined on $\mathbb{R}^+$ instead of $\mathbb{N}$. In general, $\lim\limits_{n\to\infty} \frac{g(n)}{f(n)}$ is the same as $\lim\limits_{x\to\infty} \frac{g(x)}{f(x)}$ if the second limit exists.

Exercise 0.2 *Comparison of functions part 1.*

Show that

a) $f(n) := n \log n$ grows asymptotically faster than $g(n) := n$.

b) $f(n) := n^3$ grows asymptotically faster than $g(n) := 10n^2 + 100n + 1000$.

c) $f(n) := 3^n$ grows asymptotically faster than $g(n) := 2^n$.

The following theorem can be useful to compute some limits.

Theorem 1 (L'Hôpital's rule). *Assume that functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ are differentiable, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$ and for all $x \in \mathbb{R}^+$, $g'(x) \neq 0$. If $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = C \in \mathbb{R}_0^+$ or $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \infty$, then*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} .$$

Exercise 0.3 *Comparison of functions part 2.*

Show that

a) $f(n) := n^{1.01}$ grows asymptotically faster than $g(n) := n \ln n$.

b) $f(n) := e^n$ grows asymptotically faster than $g(n) := n$.

c) $f(n) := e^n$ grows asymptotically faster than $g(n) := n^2$.

d)* $f(n) := 1.01^n$ grows asymptotically faster than $g(n) := n^{100}$.

e)* $f(n) := \log_2 n$ grows asymptotically faster than $g(n) := \log_2 \log_2 n$.

f)* $f(n) := 2^{\sqrt{\log_2 n}}$ grows asymptotically faster than $g(n) := \log_2^{100} n$.

g)* $f(n) := n^{0.01}$ grows asymptotically faster than $g(n) := 2^{\sqrt{\log_2 n}}$.

Exercise 0.4 *Simplifying expressions.*

Simplify the following expressions as much as possible without changing their asymptotic growth rates.

Concretely, for each expression $f(n)$ in the following list, find an expression $g(n)$ that is as simple as possible and that satisfies $\lim_{n\to\infty} \frac{f(n)}{g(n)} \in \mathbb{R}^+$.

It is guaranteed that all functions in this exercise take values in $\mathbb{R}^+$ (you don't have to prove it).

a) $f(n) := 5n^3 + 40n^2 + 100$

b) $f(n) := 5n + \ln n + 2n^3 + \frac{1}{n}$

c) $f(n) := n \ln n - 2n + 3n^2$

d) $f(n) := 23n + 4n \log_5 n^6 + 78\sqrt{n} - 9$

e) $f(n) := \log_2 \sqrt{n^5} + \sqrt{\log_2 n^5}$

f)* $f(n) := 2n^3 + \left(\sqrt[4]{n}\right)^{\log_5 \log_6 n} + \left(\sqrt[7]{n}\right)^{\log_8 \log_9 n}$

Exercise 0.5* *Finding the range of your bow.*

To celebrate your start at ETH, your parents gifted you a bow and (an infinite number of) arrows. You would like to determine the range of your bow, in other words how far you can shoot arrows with it. For simplicity we assume that all your arrow shots will cover exactly the same distance r, and we define r as the range of your bow. You also know that this range is at least $r \geq 1$ (meter).

You have at your disposition a ruler and a wall. You cannot directly measure the distance covered by an arrow shot (because the arrow slides some more distance on the ground after reaching distance r), so the only way you can get information about the range r is as follows. You can stand at a distance ℓ (of your choice) from the wall and shoot an arrow: if the arrow reaches the wall, you know that $\ell \leq r$, and otherwise you deduce that $\ell > r$. By performing such an experiment with various choices of the distance ℓ, you will be able to determine r with more and more accuracy. Your goal is to do so with as few arrow shots as possible.

a) What is a fast strategy to find an upper bound on the range r ? In other words, how can you find a distance $D \geq 1$ such that $r < D$, using few arrow shots ? The required number of shots might depend on the actual range r, so we will denote it by $f(r)$. Good solutions should have $f(r) \leq 10 \log_2 r$ for large values of r.

b) You are now interested in determining r up to some additive error. More precisely, you should find an estimate $\tilde{r}$ such that the range is contained in the interval $[\tilde{r} - 1, \tilde{r} + 1]$, i.e. $\tilde{r} - 1 \leq r \leq \tilde{r} + 1$. Denoting by $g(r)$ the number of shots required by your strategy, your goal is to find a strategy with $g(r) \leq 10 \log_2 r$ for all r sufficiently large.

c) Coming back to part (a), is it possible to have a significantly faster strategy (for example with $f(r) \leq 10 \log_2 \log_2 r$ for large values of r) ?

a) Prove by mathematical induction that for any positive integer n,

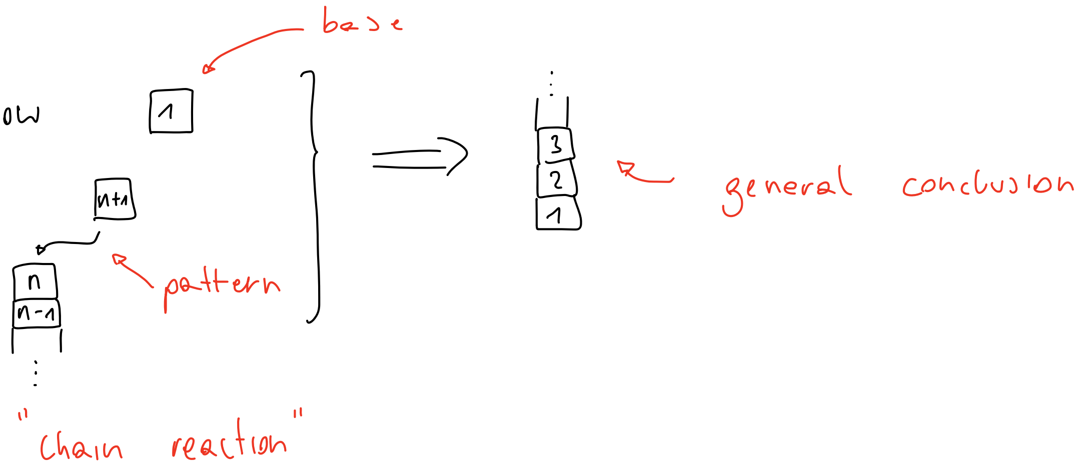
$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

What is induction? "show for all numbers?"

Intuition:

(1) We show

(2) Then



a) Prove by mathematical induction that for any positive integer n,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}. \quad (*)$$

Induction base: $n = 1$

$$1 = \frac{1(2)}{2}. \quad \checkmark$$

Induction hypothesis: Assume $(*)$ holds for some positive integer k.

(that is $1 + 2 + \ldots + k = \frac{k(k+1)}{2}$)

Induction step: $k \rightsquigarrow k+1$

$$1 + 2 + \ldots + k + (k+1) \overset{\text{Ind. hyp.}}{=} \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$\overset{?}{=}$$

Tip #1: sometimes it helps to look at "the other side":

$$\frac{(k+1)(k+2)}{2} = \frac{k^2+3k+2}{2} \quad \ldots$$

Coming back: $\quad \|$!

$$\frac{k(k+1)+2(k+1)}{2} = \frac{k^2+3k+2k}{2}.$$

! $\Rightarrow$ By the principle of mathematical induction, this is true for any positive integer n.

Tip #2: expressions like $1+2+\ldots+n$ can be confusing if a little more complex. What I like to do: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

b) **(This subtask is from August 2019 exam).** Let $T : \mathbb{N} \to \mathbb{R}$ be a function that satisfies the following two conditions:

$$T(n) \geq 4 \cdot T(\tfrac{n}{2}) + 3n \qquad \text{whenever } n \text{ is divisible by 2;}$$
$$T(1) = 4.$$

we know

Prove by mathematical induction that

$$T(n) \geq 6n^2 - 2n$$

we prove

holds whenever n is a power of 2, i.e., $n = 2^k$ with $k \in \mathbb{N}_0$.

Tip #3: read description carefully. Know what to use, what to prove, what to assume etc. "Toolset laid out on table" analogy.

k and n? two variables?

Induction by n.

IB: $k=0 \Rightarrow n = 2^0 = 1$

$T(1) = 4 \geq 6 \cdot 1 - 2$. ✓

"that property"

IH: Assume ⊛ holds for some positive integer $m = 2^k$.

[That is: $T(m) \geq 6m^2 - 2m$]

useful!

IS: $k \rightsquigarrow k+1 \Rightarrow 2^k \rightsquigarrow 2^{k+1} = 2 \cdot 2^k$

$$T(2^{k+1}) \geq 4 \cdot T(2^k) + 3 \cdot 2^{k+1}$$

$$\geq 4 \cdot (6 \cdot (2^k)^2 \ \ldots$$

complicated, easy to make mistakes here.

Instead we use m. If $n = 2^k$, then for $k+1$ we have $2m = 2^{k+1}$.

IS: $m \rightsquigarrow 2m$

$$T(2m) \geq 4\,T(m) + 3 \cdot 2 \cdot m$$

$$\overset{\text{I.H.}}{\geq} 24m^2 - 8m + 6m$$

$$= 24m^2 - 2m$$

Tip 2!

$$\geq 24m^2 - 4m$$

$$= 6 \cdot (2m)^2 - 2 \cdot (2m)$$

Tip #4: Substitute variables if necessary! less error prone.

Asymptotic Growth

When we estimate the number of elementary operations executed by algorithms, it is often useful to ignore smaller order terms, and instead focus on the asymptotic growth defined below. We denote by $\mathbb{R}^+$ the set of all (strictly) positive real numbers and by $\mathbb{R}_0^+$ the set of nonnegative real numbers.

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for $x \in \mathbb{R}$.

Exercise 0.2 *Comparison of functions part 1.*

Show that

Tip 3!

a) $f(n) := n \log n$ grows asymptotically faster than $g(n) := n$.

$$\lim_{n\to\infty} \frac{n}{n \log n} = \lim_{n\to\infty} \frac{1}{\log n} = 0.$$

b) $f(n) := n^3$ grows asymptotically faster than $g(n) := 10n^2 + 100n + 1000$.

$$\lim_{n\to\infty} \frac{10n^2 + 100n + 100}{n^3} = \lim_{n\to\infty} \left(\frac{10}{n} + \frac{100}{n^2} + \frac{100}{n^3} \right)$$

$$\xrightarrow{\text{limit rules}} \lim_{n\to\infty} \frac{10}{n} + \lim_{n\to\infty} \frac{100}{n^2} + \lim_{n\to\infty} \frac{100}{n^3}$$

read guide!

$$= 0 + 0 + 0.$$

c) $f(n) := 3^n$ grows asymptotically faster than $g(n) := 2^n$.

$$\lim_{n\to\infty} \frac{2^n}{3^n} = ?$$

what happens if we halve something an infinite number of times?

$$\xrightarrow{\text{exp. rule}} \lim_{n\to\infty} \Big(\underbrace{\tfrac{2}{3}}_{<1}\Big)^n = 0.$$

guide

Theorem 1 (L'Hôpital's rule). *Assume that functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ are differentiable, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$ and for all $x \in \mathbb{R}^+$, $g'(x) \neq 0$. If $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = C \in \mathbb{R}_0^+$ or $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \infty$, then*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

very useful! Revise deriv rules!

Exercise 0.3 *Comparison of functions part 2.*

Show that

$\log_e = \ln$.

a) $f(n) := n^{1.01}$ grows asymptotically faster than $g(n) := n \ln n$.

$$\lim_{n\to\infty} \frac{n \ln n}{n^{1.01}} = \lim_{x\to\infty} \frac{\ln x + 1}{1.01 \cdot x^{0.01}} = \dots$$

Tip #5: simplify! don't blindly apply rules/theorems

$$\lim_{n\to\infty} \frac{n \ln n}{n^{1.01}} = \lim_{n\to\infty} \frac{\ln n}{n^{0.01}} \overset{?}{=}$$

exp rule

$$\lim_{x\to\infty} \frac{\ln x}{x^{0.01}} \overset{L.H.}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{0.01\, x^{-0.99}} = \lim_{x\to\infty} \frac{1}{0.01\, x^{0.01}} = 0.$$

why x anyways? Analysis on $\mathbb{R}$, limits, diff, cont. ...

$\mathbb{N} \subseteq \mathbb{R}$.

b) $f(n) := e^n$ grows asymptotically faster than $g(n) := n$.

$$\lim_{x\to\infty} \frac{x}{e^x} \overset{\text{L.H.}}{=} \lim_{x\to\infty} \frac{1}{e^x} = 0.$$

Tip 3.

c) $f(n) := e^n$ grows asymptotically faster than $g(n) := n^2$.

$$\lim_{x\to\infty} \frac{x^2}{e^x} \overset{\text{L.H.}}{=} \lim_{n\to\infty} \frac{2x}{e^x} \overset{\text{L.H.}}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

$$\xRightarrow{\text{lim rules}} \underbrace{2}_{c} \lim_{x\to\infty} \frac{1}{e^x} = 2 \cdot 0$$

guide

can be any constant if independent of x

d)* $f(n) := 1.01^n$ grows asymptotically faster than $g(n) := n^{100}$.

$$\lim_{x\to\infty} \frac{x^{100}}{1.01^x} = ?$$

n base, n exponent? What now?

$e^{\ln}$-Trick $\Longrightarrow$ $x^{100} = e^{\ln x^{100}}$, $1.01^x = e^{\ln 1.01^x}$

$$= e^{100 \ln x} \qquad = e^{x \ln 1.01}$$

guide (+ log rules) Revise!

? compare exponents

$$\lim_{x\to\infty} \frac{e^{100 \ln x}}{e^{x \ln 1.01}} = \left[\lim_{x\to\infty} e^{100 \ln x - x \ln 1.01}\right]$$

we look at exponent!

lim rules $\Longrightarrow$ $\lim\limits_{x\to\infty} e^{100 \ln x - x \ln 1.01}$

guide

$$= e^{\lim\limits_{x\to\infty} 100 \ln x - x \ln 1.01}$$

$$\lim_{x\to\infty} 100 \ln x - x \ln 1.01$$

$$= \lim x \left(100 \underbrace{\frac{\ln x}{x}}_{\xrightarrow{\infty} 0} - \ln 1.01\right)$$

$$= -\infty$$

shall show that exp always faster than power.

$\Longrightarrow$ $e^{-\infty} = 0$, thus $\lim\limits_{x\to\infty} \dfrac{x^{100}}{(1.01)^x} = 0$.

e)* $f(n) := \log_2 n$ grows asymptotically faster than $g(n) := \log_2 \log_2 n$.

$$\lim_{x\to\infty} \frac{\log_2 \log_2 x}{\log_2 x} \overset{\text{L.H.}}{=} \lim_{x\to\infty} \frac{\overbrace{\frac{1}{\log_2 x}\cdot\frac{1}{x}}^{\text{chain rule}}}{\frac{1}{x}}$$

different than sol.

$$= \lim_{x\to\infty} \frac{x}{x \log_2 x} \overset{\text{ex. 0.2 (a.)}}{=} 0 .$$

f)* $f(n) := 2^{\sqrt{\log_2 n}}$ grows asymptotically faster than $g(n) := \log_2^{100} n$. $= (\log_2 n)^{100}$

$$\lim_{x\to\infty} \frac{\log_2^{100} x}{2^{\sqrt{\log_2 x}}}$$

because we try to match the denominator

$e^{\ln x}$-Trick (with $2^{\log_2 x}$)

$\Longrightarrow$ $\log_2^{100} x = 2^{100 \log_2 \log_2 x}$

$$\lim_{x\to\infty} \frac{\log_2^{100} x}{2^{\sqrt{\log_2 x}}} = \lim_{x\to\infty} \frac{2^{100 \log_2 \log_2 x}}{2^{\sqrt{\log_2 x}}}$$

$$= \lim_{x\to\infty} 2^{100 \log_2 \log_2 x - \sqrt{\log_2 x}}$$

we look at the exponent:

$$\lim_{x\to\infty} 100 \log_2 \log_2 x - \sqrt{\log_2 x}$$

$$= \lim_{x\to\infty} \log_2 x \left(\frac{100 \log_2\log_2 x}{\sqrt{\log_2 x}} - 1 \right)$$

$$\lim_{n\to\infty}\left(100\log_2\log_2 n - \sqrt{\log_2 n}\right) = \lim_{n\to\infty}\left(-\sqrt{\log_2 n}\left(1 - 100\frac{\log_2\log_2 n}{\sqrt{\log_2 n}}\right)\right) = -\infty\,.$$

$\frac{\infty}{\infty} \longrightarrow 0$???

solutions

instead $\Longrightarrow$

$$\log_2 x = \frac{\ln x}{\ln 2}$$

guide.

Thus $$\log_2 \log_2 x = \frac{\ln\left(\frac{\ln x}{\ln 2}\right)}{\ln(2)}$$

$$\frac{d}{dx}\frac{\ln\left(\frac{\ln x}{\ln 2}\right)}{\ln 2} = \frac{1}{\ln 2}\left(\frac{d}{dx}\ln\left(\frac{\ln x}{\ln 2}\right)\right)$$

$$= \frac{1}{\ln 2}\left(\frac{\ln 2}{\ln x}\cdot\frac{1}{\ln 2\, x}\right) = \frac{1}{\ln(2)\, x \ln x}.$$

$$\sqrt{\log_2 x} = \sqrt{\frac{\ln x}{\ln 2}}, \quad \frac{d}{dx}\sqrt{\frac{\ln x}{\ln 2}} = \frac{1}{2\sqrt{\frac{\ln x}{\ln 2}}}\cdot\frac{1}{\ln(2)x}$$

$$= \frac{1}{2\sqrt{\ln(2)}\, x \sqrt{\ln x}}$$

$$\lim_{x\to\infty} \frac{\log_2 \log_2 x}{\sqrt{\log_2 x}} \overset{L.H.}{=} \lim_{x\to\infty} \frac{\frac{1}{\ln(2)\, x \ln x}}{\frac{1}{2\sqrt{\ln(2)}\, x \sqrt{\ln x}}}$$

$$= \lim_{x\to\infty} \frac{2}{\sqrt{\ln(2)}\sqrt{\ln x}} = \frac{2}{\sqrt{\ln(2)}} \lim_{x\to\infty} \frac{1}{\sqrt{\ln x}} = 0$$

$$\Longrightarrow \quad \lim_{x\to\infty} \log_2 x \left(\frac{100 \log_2 \log_2 x}{\sqrt{\log_2 x}} - 1 \right)$$

$$= -\infty$$

$$\Longrightarrow \quad \lim_{x\to\infty} \frac{\log_2^{100} x}{2^{\sqrt{\log_2 x}}}$$

$$= \lim_{x\to\infty} 2^{100 \log_2 \log_2 x - \sqrt{\log_2 x}} = 0$$

g)* $f(n) := n^{0.01}$ grows asymptotically faster than $g(n) := 2^{\sqrt{\log_2 n}}$.

$$\lim_{x \to \infty} \frac{2^{\sqrt{\log_2 x}}}{x^{0.01}} = \lim_{x \to \infty} \frac{2^{\sqrt{\log_2 x}}}{2^{0.01 \log_2 x}}$$

$$= \lim_{x \to \infty} 2^{\sqrt{\log_2 x} - 0.01 \log_2 x}$$

we look at the exponent:

$$\lim_{x \to \infty} \sqrt{\log_2 x} - 0.01 \log_2 x$$

$$= \lim_{x \to \infty} 0.01 \log_2 x \left(\underbrace{\frac{\sqrt{\log_2 x}}{0.01 \log_2 x}}_{= \frac{1}{0.01 \sqrt{\log_2 x}} \xrightarrow{\infty} 0} - 1 \right)$$

$$= -\infty$$

Thus

$$\lim_{x \to \infty} \frac{2^{\sqrt{\log_2 x}}}{x^{0.01}} = \lim_{x \to \infty} 2^{\sqrt{\log_2 x} - 0.01 \log_2 x} = 0.$$

Exercise 0.4 *Simplifying expressions.*

Simplify the following expressions as much as possible without changing their asymptotic growth rates.

Concretely, for each expression $f(n)$ in the following list, find an expression $g(n)$ that is as simple as possible and that satisfies $\lim_{n\to\infty} \frac{f(n)}{g(n)} \in \mathbb{R}^+$.

It is guaranteed that all functions in this exercise take values in $\mathbb{R}^+$ (you don't have to prove it).

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$g(n) = n^3$ ...

b) $f(n) := 5n + \ln n + 2n^3 + \frac{1}{n}$

$g(n) = n^3$ ...

c) $f(n) := n \ln n - 2n + 3n^2$

$g(n) = n^2$ ...

d) $f(n) := 23n + 4n \log_5 n^6 + 78\sqrt{n} - 9$

$n \log n$ or $\sqrt{n}$? test it : $\lim \frac{\sqrt{n}}{n \ln n}$ ...

$\tilde{g}(n) = 4n \log_5 n^6 = 24\, n \log_5 n$

$= \underbrace{\frac{24}{\ln 5}}_{c}\, n \ln n$

$g(n) = n \ln n$

$\Longrightarrow \frac{24}{\ln 5}$.

f)* $f(n) := 2n^3 + \left(\sqrt[4]{n}\right)^{\log_5 \log_6 n} + \left(\sqrt[7]{n}\right)^{\log_8 \log_9 n}$

$\left(\sqrt[4]{n}\right)^{\log_5 \log_6 n}$ or $\sqrt[7]{n}^{\log_8 \log_9 n}$?

Test it !

$$\lim_{n\to\infty} \frac{\sqrt[7]{n}^{\log_8 \log_9 n}}{\sqrt[4]{n}^{\log_5 \log_6 n}}$$

intuition : the bigger exp the faster growing.

$$= \lim_{n\to\infty} n^{\frac{1}{7}\log_8 \log_9 n - \frac{1}{4}\log_5 \log_6 n}$$

$$\xRightarrow{\text{exponent}} \lim_{n\to\infty} \frac{1}{7}\log_8 \log_9 n - \frac{1}{4}\log_5 \log_6 n$$

Problem: same growing speed, all logs !

but still $\lim_{n\to\infty} * = -\infty$ since

$\log_a x \leq \log_b y$ if $x \leq y$ and $a \geq b$.

Thus $\xRightarrow{\infty} -\infty$

$$\lim_{n\to\infty} n^{\frac{1}{7}\log_8 \log_9 n - \frac{1}{4}\log_5 \log_6 n} = 0.$$

Moreover, we also have

$$\lim_{n\to\infty} \frac{2n^3}{\left(\sqrt[4]{n}\right)^{\log_5 \log_6 n}} = 2 \lim_{n\to\infty} n^{3-\frac{1}{4}\log_5 \log_6 n} = 0\,.$$

Let $g(n) := n^{\frac{1}{4}\log_5 \log_6 n}$. Then indeed we have

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1 \in \mathbb{R}^+.$$

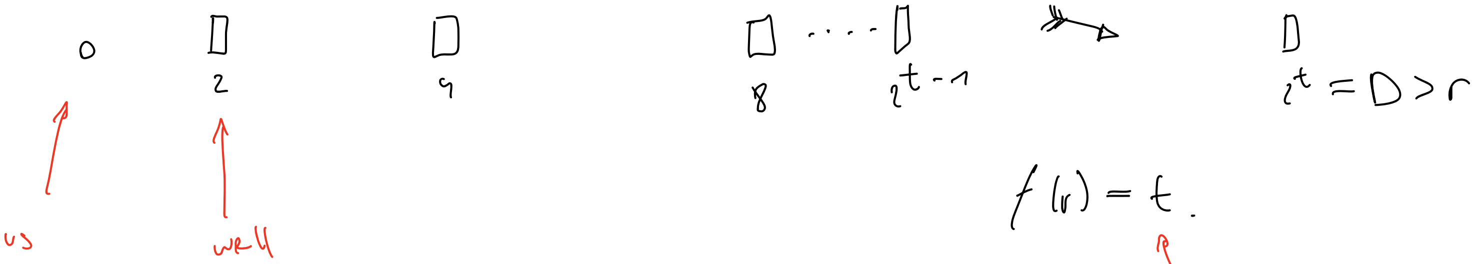
Exercise 0.5* *Finding the range of your bow.*

To celebrate your start at ETH, your parents gifted you a bow and (an infinite number of) arrows. You would like to determine the range of your bow, in other words how far you can shoot arrows with it. For simplicity we assume that all your arrow shots will cover exactly the same distance r, and we define r as the range of your bow. You also know that this range is at least $r \geq 1$ (meter).

You have at your disposition a ruler and a wall. You cannot directly measure the distance covered by an arrow shot (because the arrow slides some more distance on the ground after reaching distance r), so the only way you can get information about the range r is as follows. You can stand at a distance ℓ (of your choice) from the wall and shoot an arrow: if the arrow reaches the wall, you know that $\ell \leq r$, and otherwise you deduce that $\ell > r$. By performing such an experiment with various choices of the distance ℓ, you will be able to determine r with more and more accuracy. Your goal is to do so with as few arrow shots as possible.

a) What is a fast strategy to find an upper bound on the range r ? In other words, how can you find a distance $D \geq 1$ such that $r < D$, using few arrow shots ? The required number of shots might depend on the actual range r, so we will denote it by $f(r)$. Good solutions should have $f(r) \leq \boxed{10 \log_2 r}$ for large values of r.

Tip # 6: hidden information



$f(r) = t.$

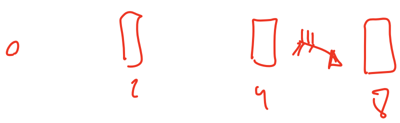
\# shots

$f(r) = \lceil \log_2 r \rceil \leq 1 + \log_2 r \leq 10 \log_2 r$

$\Longrightarrow 2^{1/9} \leq r.$

what is ceil? assume $r = 5$.

$\log_2 5 \approx 2.32$



we can't say throws! we have to shoot $\lceil \rceil$ times.

"This strategy finds D for $r \geq 2^{1/9}$ in $\lceil \log_2 r \rceil$ shots."

perfect

b) You are now interested in determining r up to some additive error. More precisely, you should find an estimate $\tilde{r}$ such that the range is contained in the interval $[\tilde{r}-1, \tilde{r}+1]$, i.e. $\tilde{r}-1 \leq r \leq \tilde{r}+1$. Denoting by $g(r)$ the number of shots required by your strategy, your goal is to find a strategy with $g(r) \leq 10 \log_2 r$ for all r sufficiently large.

we again use strat from (a).

$2^{t-1} = \frac{D}{2}$ $\qquad$ $2^t = D$

$\Rightarrow r \in [D/2, D]$ in $f(r) = \lceil \log_2 r \rceil$ shots.

Now we position ourselves in the middle:

$2^{t-1} = \frac{D}{2}$ $\qquad$ $\frac{(D/2 + D)}{2} = \frac{3}{4}D$ $\qquad$ $2^t = D$

$[D/2, \frac{3}{4}D][\frac{3}{4}D, D]$

two intervals

now we shoot from the middle and see if reaches the wall:

- Yes, $r \in [\frac{3}{4}D, D]$
- No, $r \in [\frac{1}{2}D, \frac{3}{4}D]$

Why? What if $r < D - \frac{3}{4}D = \frac{1}{4}D$

repeat until $[a,b]$ such that $b - a \leq 2$

then $\tilde{r} = (a+b)/2$.

$\underbrace{|\quad}_{\leq 1} \tilde{r} \underbrace{\quad|}_{\leq 1}$

shots ?

- $f(r) = \lceil \log_2 r \rceil$ from finding D

- halving intervals until $b - a \leq 2$?

  we start with $\underbrace{[\frac{1}{2}D, D]}_{\frac{D}{2}} \to \frac{D}{4} \to \frac{D}{8} \ldots$

  t' times $\Rightarrow$ $t' = \max\{\underbrace{\log_2 D - 2}_{t}, \underbrace{0}_{D=2}\}$

negative shots ?

$\frac{D}{2^{t'+1}} \leq 2 \Rightarrow D \leq 2^{t'+2}$

we start with $\frac{D}{2}$

$\log_2 D - 2 \leq t^2$

(We skip c.)